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ROY DUBISCH, *Editor*

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NOTES ON PSEUDO-RECTANGLES

J. ALLARD, University of Sherbrooke, Canada

A brief survey of geometry texts and scientific dictionaries shows that parallels can have many definitions that are more or less equivalent. Some definitions are simple like: "Two straight lines which lie in a plane but do not meet are parallel." Concentric circles are parallel; to apply the above definition, the circles must be visualized as an infinite number of infinitely small straight lines that are parallel to each other. A definition that was popular a 100 years ago is: "A curve obtained by measuring from a conic on each normal a constant distance is a parallel." In a recent book on curves we find: "While every curve has but one evolute, it has many involutes; for the initial point, where the involute cuts the original curve, may be chosen arbitrarily. The various curves so obtained are called parallel curves." This definition is a rigorous way of saying that two curves or lines are parallel if they have the same slope.

Parametric equations of parallels to pseudo-rectangles can be obtained with the method of envelopes applied to the equation

$$\sum_{i=1}^{2}\left[\left(\frac{x_i}{a_i}\right)^2\right]^n = 1, \tag{1}$$

which yields the circle, the square [1], the ellipse, and the rectangle as particular cases. When $a_1 = a$, $a_2 = b$, $x_1 = x$, $x_2 = y$ we have

$$\left[\left(\frac{x}{a}\right)^2\right]^n + \left[\left(\frac{y}{b}\right)^2\right]^n = 1, \tag{2}$$

a and b being semi-major and semi-minor axes and n a parameter. It can be shown from elementary calculus that the area A of these pseudo-rectangles is

$$A = \frac{4ab\{\Gamma(1/2n + 1)\}^2}{\Gamma(1/n + 1)}. \tag{3}$$

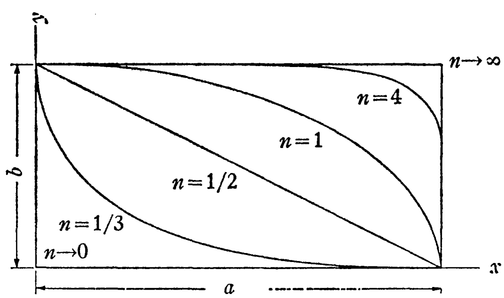

FIG. 1.

Let Fig. 1 represent the first quadrant of

$$\left[\left(\frac{x}{a}\right)^2\right]^n + \left[\left(\frac{y}{b}\right)^2\right]^n = 1.$$

When $n \to 0$ the locus of (2) tends to a "rectangular" hypocycloid with a null area; when $n = \frac{1}{2}$ a $2a$ by $2b$ rhombus is obtained; when $n = 1$ an ellipse is obtained; and when $1 < n < \infty$, "elliptic type curves" [2] or higher plane curves are obtained which I call pseudo-rectangular curves, because when $n \to \infty$ a geometrical rectangle is obtained.

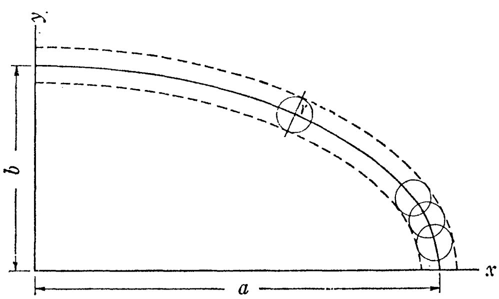

FIG. 2.

Let us consider a family of circles of constant radius r with their centers lying on the locus of (2) as shown in Fig. 2. The two loci of the end points of all normals to (2) of length r are called the envelopes of the family of circles and are parallel to (2) according to our definition of a parallel curve.

Let the equation of the family of circles be

$$(x - \alpha)^2 + (y - \beta)^2 = r^2, \tag{4}$$

and the equation of the basic pseudo-rectangle be

$$\left(\frac{\alpha}{a}\right)^{2n} + \left(\frac{\beta}{b}\right)^{2n} = 1. \tag{5}$$

From the method of envelopes the parametric equations of the envelope or parallel curve are of the form

$$x = x(\alpha, a, b, r), \qquad y = y(\beta, a, b, r), \tag{6}$$

where a and b are kept constant for a particular family of pseudo-rectangles and r may have a maximum value. When we use (5), equation (4) becomes

$$(x - \alpha)^2 + \left\{y - b\left[1 - \left(\frac{\alpha}{a}\right)^{2n}\right]^{1/2n}\right\}^2 - r^2 = 0. \tag{7}$$

Differentiating (7) partially with respect to α, we obtain

$$x - \alpha = \pm \frac{\frac{rb}{a}\,\alpha^{2n-1}(a^{2n} - \alpha^{2n})^{1/2n-1}}{\left[1 + \frac{b^2}{a^2}\,\alpha^{4n-2}(a^{2n} - \alpha^{2n})^{1/n-2}\right]^{1/2}}, \tag{8}$$

which can be written more conveniently in the form

$$x = \alpha \pm \frac{rb\alpha^{2n-1}}{\left[a^2(a^{2n} - \alpha^{2n})^{2-1/n} + b^2\alpha^{4n-2}\right]^{1/2}}; \tag{9}$$

by a similar method, the y coordinate of the parallel is

$$y = \beta \pm \frac{ra\beta^{2n-1}}{\left[b^2(b^{2n} - \beta^{2n})^{2-1/n} + a^2\beta^{4n-2}\right]^{1/2}}, \tag{10}$$

where β and α are related by (5).

Particular cases of interest are:

1. When $n=1$ the "elliptic race track" problem is obtained and the parametric formula for x on the inner boundary of the elliptic race track is:

$$x = \alpha\left[1 - \frac{rb}{(a^4 - \alpha^2a^2 + b^2\alpha^2)^{1/2}}\right] = \alpha\left[1 - \frac{rb}{(a^4 - \alpha^2a^2e^2)^{1/2}}\right], \tag{11}$$

where

$$e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}.$$

A mechanical method can be also used to plot these parallels to ellipses [**3**].

2. When $n \to \infty$, a geometrical rectangle is approached and $\alpha \to a$, or $\beta \to b$, which reduces formula (9) to $x = a \pm r$.

3. When $n = \frac{1}{2}$ a family of rhombuses is obtained and their parametric equations are

$$x = \alpha \pm \frac{rb}{(a^2 + b^2)^{1/2}}, \tag{12}$$

$$y = \beta \pm \frac{ra}{(a^2 + b^2)^{1/2}}. \tag{13}$$

Using the fact that $[(\alpha/a)^2]^{1/2} + [(\beta/b)^2]^{1/2} = 1$ we get

$$b(x^2)^{1/2} + a(y^2)^{1/2} \mp r(b^2 + a^2)^{1/2} - ab = 0, \tag{14}$$

an equation for a family of rhombuses.

In the above examples there is a maximum value that r can take when the minus signs are used in (12) and (13). That is, r cannot exceed the minimum radius of curvature of the pseudo-rectangle. By elementary calculus, it can be shown that the radius of curvature R of (2) is

$$R = \frac{\left[a^2b^{4n-2}(a^{2n} - x^{2n})^{2-1/n} + b^{4n}x^{4n-2}\right]^{3/2}}{(2n - 1)a^{2n+2}b^{6n-2}x^{2n-2}(a^{2n} - x^{2n})^{(n-1)/n}}. \tag{15}$$

For an extremum we have $dR/dx=0$; this leads to the formula for the value x' of x for which the extremum is obtained.

Substituting this value of x' in (15), we obtain the minimum radius of curvature of the pseudo-rectangle.

These higher plane curves have other interesting properties and uses [4, 5, 6]. Pseudo-rectangles sometimes called storoids were popular topics at the end of the 19th century and they were treated by Lamé, Dirichlet and Minkowski. Today, analytic geometry texts seldom discuss these curves. In a few applied mathematics texts, they are briefly treated as a by-product of Dirichlet's multiple integrals.

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USING "CROSS PRODUCTS" TO DERIVE CRAMER'S RULE

DONALD R. BARR, Colorado State University

Introduction. Frequently in mathematics one encounters situations in which two different theorems or techniques can be used to solve a certain class of problems. When this happens, we are tempted to ask whether these theorems have a connection stronger than that of being applicable to a common class of problems. In fact, we might wonder if it is possible to prove one theorem from the other through this class of problems.

An example leading to such a proof is given in what follows. The problem is that of finding the equation of a plane, given three of its points.

The cartesian equation of a plane in E^3 (three-dimensional Euclidean space) can be put in the form

$$\alpha x+\beta y+\gamma z=1, \tag{1}$$

where α, β, and γ are real numbers. The plane is determined by three noncollinear points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) in it. Given these points, then, we are able to determine the coefficients in equation (1) in the following two ways:

For an extremum we have $dR/dx = 0$; this leads to the formula for the value x' of x for which the extremum is obtained.

Substituting this value of x' in (15), we obtain the minimum radius of curvature of the pseudo-rectangle.

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A. *Algebraic Approach.* Since the three points must satisfy equation (1), we have a system of three linear equations in the unknowns α, β, and γ. This system can be written

$$A\mathbf{c}^T = \mathbf{1}^T \tag{2}$$

where A is a matrix whose i-th row is composed of the coordinates of the i-th given point, $\mathbf{c}=(\alpha, \beta, \gamma)$, $\mathbf{1}=(1, 1, 1)$, and the "T" indicates "transpose." Of course, this system can be solved using Cramer's Rule.

G. *Geometric Approach.* A vector normal to the plane can be obtained by taking the cross product (also called the vector product or outer product) of two nonzero, nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$ in the plane. Since the coefficient vector $\mathbf{c}$ is also normal to the plane, it follows that $\mathbf{u}\times\mathbf{v}=\lambda\mathbf{c}$, where λ is a real number. Knowing $\lambda\mathbf{c}$, we can easily obtain the coefficients in equation (1).

The observation that (A) and (G) both lead to the solution of equation (2) suggests the possibility of proving Cramer's Rule using cross products. (Note that any system of three nonhomogeneous linear equations in three unknowns may be put in the form of equation (2).)

Case for $\mathbf{E}^3$. Suppose (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are three noncollinear points on a plane whose equation is given by (1). We seek the solution to the system (2). The vectors $\mathbf{u}=(x_2-x_1)\mathbf{i}+(y_2-y_1)\mathbf{j}+(z_2-z_1)\mathbf{k}$ and $\mathbf{v}=(x_3-x_1)\mathbf{i}+(y_3-y_1)\mathbf{j}+(z_3-z_1)\mathbf{k}$ are in the plane, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the usual orthogonal unit basis vectors for E^3. Thus, using the usual determinant expansion for cross products, we have that

$$\mathbf{u}\times\mathbf{v} = \lambda\alpha\mathbf{i} + \lambda\beta\mathbf{j} + \lambda\gamma\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} \tag{3}$$

is a vector normal to the plane.

Multiplying equation (1) through by λ and equating multiples of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in equation (3), we have

$$\begin{aligned}[(y_2-y_1)(z_3-z_1)-(y_3-y_1)(z_2-z_1)]x+[(x_3-x_1)(z_2-z_1)-(x_2-x_1)(z_3-z_1)]y \\ + [(x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1)]z=\lambda,\end{aligned} \tag{4}$$

where λ is obtained by letting $x=x_1$, $y=y_1$, and $z=z_1$ in equation (4). It is easily verified by expanding the respective determinants that equation (4) may be written as

$$|A_1|x + |A_2|y + |A_3|z = |A|, \tag{5}$$

where $|A_j|$ is the same as $|A|$ except that the j-th column has been replaced by a column of ones.

Comparing equations (1) and (5) we have $\alpha=|A_1|/|A|$ if $|A|\neq 0$, with similar expressions for β and γ. This is, of course, Cramer's Rule for the system (2). A geometrical condition on the plane which insures that system (2) is nonhomogeneous and has nonzero coefficient determinant is that it does not contain the origin; i.e., it has nonzero intercepts.

Special case for E^2 and generalizations. The argument used above is easily specialized to E^2 and generalized to higher dimensional spaces, provided that we merely replace the cross product arising in E^3 by a determinant of appropriate order. These cases are also interesting in that they point out a "cross product" technique of finding a vector in E^n normal to a given $n-1$ flat. One can easily prove this normality using the fact that determinants with one or more repeated rows are zero. It is also easy to show that the geometrical condition on the flat insuring the existence of the nonhomogeneous system and nonzero coefficient determinant remains the same under change in dimensionality of the space considered.

TANNERY'S THEOREM

R. P. Boas, Jr., Northwestern University

The title of this note is presumably as unfamiliar to most American mathematicians as it was to me when I encountered it recently. Tannery's theorem for series ([1], p. 123; [2], p. 136) deals with limits such as $\lim_{n\to\infty} (1+x/n)^n$ and ([4], p. 467)

$$\lim_{n\to\infty} \left\{ \left(\frac{n}{n}\right)^n + \left(\frac{n-1}{n}\right)^n + \cdots + \left(\frac{1}{n}\right)^n \right\},$$

and even with a derivation of the power series for the sine and cosine without using Taylor's formula. It says that if $f_k(n) \to L_k$ for each k, as $n\to\infty$, and if $|f_k(n)| \leq M_k$ with $\sum M_k$ convergent then

$$f_1(n) + f_2(n) + \cdots + f_p(n) \to \sum_1^\infty L_k,$$

provided that $p\to\infty$ as $n\to\infty$. Bromwich remarks ([2], p. 136), " . . . the test for the theorem is substantially the same as the M-test due to Weierstrass . . . . The proof, too, is almost the same." It is a good test of a student's grasp of uniform convergence to ask him to verify that the analogy here is extremely close: the theorem is a special case of the M-test. (Cf. [3], p. 122.)

There are similar theorems for infinite products and for improper integrals.

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THE THREE FACTORY PROBLEM

IRWIN GREENBERG, New York University and RAYMOND A. ROBERTELLO, Fairleigh-Dickinson University

The problem. The three factory problem can be stated as follows: a company has three factories, U, V, and W, which must receive some commodity in lots of u, v, and w respectively, each receiving period. The factories are to be supplied from a single warehouse. Assuming that shipping costs are proportional to the product of quantity shipped and distance, where should the warehouse be located so that the sum of the shipping costs is minimized? Mathematically, the problem is to find the point Z such that (see Figure 1)

$$u\overline{UZ} + v\overline{VZ} + w\overline{WZ} = \text{minimum}, \quad \text{where} \quad u, v, w > 0.$$

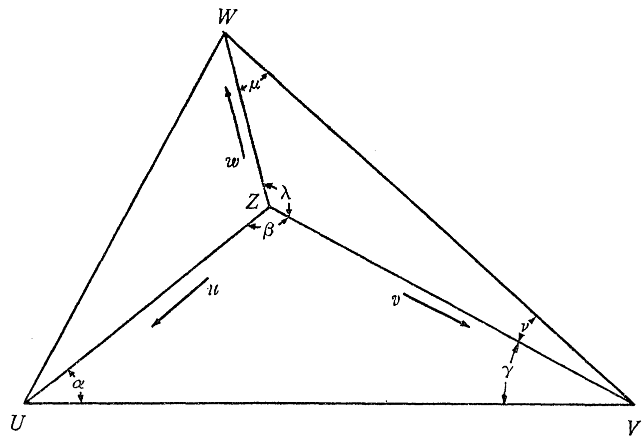

FIG. 1. The three factory problem.

The solution. *Case* 1. U, V, and W are collinear, with V between U and W. In this case the solution always lies at V unless either $u > v+w$ (in which case U is the solution) or $w > v+u$ (in which case W is the solution).

If $u = v+w$, all points in the closed interval $[U, V]$ are solutions. If $w = v+u$, then all points in $[V, W]$ are solutions.

Case 2. U, V, and W form a triangle. In this case the solution lies either within the triangle or at the factory which receives the largest shipment. If Z lies within the triangle UVW, then the situation is shown in Figure 1. The interior angles of the triangle are all less than π and they sum to 2π. Thus, at least two of the central angles will be obtuse. One of these, β, can be written in the form

$$\beta = \frac{\pi}{2} + a, \qquad 0 < a < \frac{\pi}{2}, \tag{1}$$

where

$$\sin a = \frac{u^2 + v^2 - w^2}{2uv}. \tag{2}$$

The condition $0<a<\pi/2$ implies $0<\sin a<1$, which implies that $|u-v|<w$. This and the corresponding constraints derived from the other two interior angles lead to the requirements

$$u < v + w; \qquad v < u + w; \qquad w < u + v. \tag{3}$$

If the quantity required at any factory is greater than the sum of the quantities required at the other two, then the warehouse should be located at the factory with the largest requirement.

The other requirement for Z to lie in the triangle is that β be greater than the angle at W. If this condition is not met, then the solution will lie at point W. (Since the labelling of U, V, and W is arbitrary, the same restriction applies to U and V and their corresponding interior angles.) A necessary but not sufficient condition for this to occur is $w \geqq \max(u, v)$.

When Z lies in the triangle, its location can be determined by specifying the angles α and γ, both of which are acute:

$$\tan \alpha = \frac{\overline{WV}(\cos a) \sin\left[a + c + 2\tan^{-1}\left(\frac{R}{S - \overline{WU}}\right)\right]}{\overline{UV}\cos c - \overline{WV}(\cos a)\cos\left[a + c + 2\tan^{-1}\left(\frac{R}{S - \overline{WU}}\right)\right]} \tag{4}$$

$$\gamma = \frac{\pi}{2} - a - \alpha, \tag{5}$$

where a is defined by equation (2), and c, S, and R are defined by

$$\sin c = \frac{w^2 + v^2 - u^2}{2wv}, \tag{6}$$

$$S = \tfrac{1}{2}(\overline{UV} + \overline{WV} + \overline{WU}), \tag{7}$$

$$R = \left[\frac{(S - \overline{UV})(S - \overline{WV})(S - \overline{WU})}{S}\right]^{1/2}. \tag{8}$$

If $u=v=w$ and the solution does not lie in the triangle, it lies at the vertex of the triangle with the largest angle. If $u=v>w$ and the solution is not in the triangle, it is at point U or V, whichever has the larger angle.

Discussion. The derivation of the solutions which can be accomplished using only trigonometry and analytic geometry, is quite long and tedious but involves generally well-known principles. For these reasons, the derivation will not be presented in its entirety, but merely outlined. It is anticipated that the interested reader can easily fill in the missing steps, should he so desire.

Case 1. The solution can be verified by moving a distance δ away from the indicated optimum and comparing the total costs. For example, assume that $v>u+w$. The solution lies at the point V and the total cost is $u\,\overline{UV}+w\,\overline{VW}$. Moving a distance $\delta > 0$ towards U gives a total cost of $u(\overline{UV} - \delta) + v\delta + w(\overline{VW}+\delta)$. Now

$$u(\overline{UV} - \delta) + v\delta + w(\overline{VW} + \delta) = u\overline{UV} + \delta(v + w - u) + w\overline{VW}.$$

Since $v > u + w$ implies $v + w > u$, then the new cost is larger than the original. The same argument holds for moving a distance δ toward W. Thus V gives the lowest cost.

The same kind of argument can be used to verify the solutions for any other relations between u, v, and w.

Case 2. Consider the locus of all points P such that

$$ur_U + vr_V = \text{constant}, \tag{9}$$

where r_U is the distance from the P to U and r_V is the distance from P to V. This locus is an oval of Descartes (also called a cartesian oval). Actually, the oval of Descartes consists of two ovals and is the locus of all points P such that

$$\pm\, ur_U \pm vr_V = \text{constant}.$$

The inner oval is the locus of (9); its equation, in polar coordinates with the origin at U, can be written

$$r = \frac{ku - v^2\overline{UV}\cos\theta - v\sqrt{\{k^2 + (\overline{UV})^2u^2 - 2k\overline{UV}u\cos\theta - v^2(\overline{UV})^2\sin^2\theta\}}}{u^2 - v^2},$$

where k is the constant of equation (9).

If the negative sign before the radical is replaced by a plus, the outer oval is obtained. The inner oval has no multiple points. For a full discussion of the properties of these ovals see H. Hilton, *Plane Algebraic Curves*, Oxford University Press, 1920, page 319.

In particular, consider the locus which passes through the solution point Z, i.e., the constant is $u\,\overline{UZ} + v\,\overline{VZ}$. It is clear that the point W must lie either on this oval or outside it. This can be seen by assuming that the locus encloses W, and then considering the cartesian oval which passes through W. Since this latter curve is contained within the former, choosing the point W gives a smaller value for the sum of the weighted distances to U and V as well as setting the shipping cost to W equal to zero. Thus, W cannot be inside the locus for U and V which passes through the optimum. Assume that W does not lie on the locus. Then the solution, Z, is the point on the curve closest to W; that is, since there are no multiple points, where the tangent to the curve is perpendicular to the line drawn from that point to W.

Applying the same arguments to the loci for U and W and for V and W, one obtains the situation shown in Figure 2, where T_{UV} is tangent to the cartesian oval for U and V and is perpendicular to WZ; T_{VW} is tangent to the cartesian oval for V and W and is perpendicular to UZ; and T_{UW} is tangent to the cartesian oval for U and W and is perpendicular to VZ. From elementary trigonometric considerations, it can be seen that

$$\theta_1 = \psi_2 \equiv a, \qquad \Phi_1 = \theta_2 \equiv b, \qquad \psi_1 = \Phi_2 \equiv c, \quad \text{and} \quad a + b + c = \pi/2. \tag{10}$$

Assuming that each locus can be expressed in terms of arc length, t (a parameter), then at any point along the locus for U and V we can make the classical assumption that the ratio of a small chord to its arc is very nearly equal to unity and that the limit approaches one as the arc length approaches zero. This leads to

$$\cos \Phi_1 = \frac{-dr_U}{dt}, \qquad \cos \Phi_2 = \frac{dr_V}{dt}, \tag{11}$$

the usual representation of an angle between a tangent and an axis in terms of derivatives with respect to arc length.

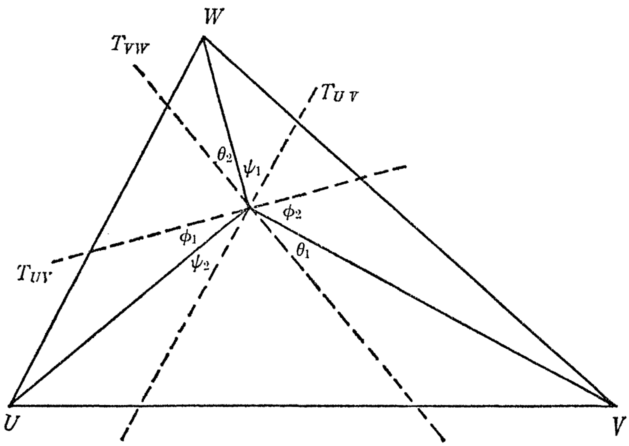

FIG. 2. Trigonometry of the three factory problem.

If equation (9) is differentiated with respect to t, and equations (10) and (11) introduced into the result; and if the procedure is repeated for the other two ovals, one eventually obtains equation (2) for sin a, equation (6) for sin c, and

$$\sin b = \frac{u^2 + w^2 - v^2}{2uw} \cdot \tag{12}$$

Equation (1) can be obtained from equation (10) and Figure 2; notice that

$$\beta = \theta_1 + \theta_2 + \psi_1 + \psi_2 = 2a + b + c = a + \pi/2.$$

Equations (4) and (5) can be derived as follows: referring to Figure 1, apply the law of sines to triangles VWZ and UVZ. Solve the first expression for $\overline{VZ}$ in terms of $\overline{VW}$ and the appropriate angles and solve the second for $\overline{VZ}$ in terms of $\overline{UV}$ and the angles. Equating the two yields an expression involving α, β, $\overline{UV}$, $\overline{VW}$, λ, and μ. Now λ can be shown to equal $\frac{1}{2}\pi+c$ in the same manner as equation (1) was proved. Also, μ can be eliminated by making use of the fact that the angles of triangles UVZ and VWZ must each add to π, and of the relationship

$$\gamma + \nu = 2 \tan^{-1}\left[\frac{R}{S - \overline{UW}}\right].$$

After some tedious manipulation, equation (4) is obtained, and then equation (5).

Thus, if the solution does not lie at a vertex, its location can be determined from (4) and (5).

The derivation of (4) and (5) tacitly assumes that Z lies in the triangle UVW. To show that it cannot lie outside the triangle one can construct a perpendicular from any point outside the triangle to the nearest side and show that at the intersection, the total cost of shipping is lower than the cost of shipping from the exterior point. (If the intersection of the perpendicular and the side lies outside the triangle along the *extension* of the side, one should move to the nearest vertex to demonstrate that the cost of shipping is lower.) One can also show that if the solution lies on a side of the triangle, it must lie at a vertex. This follows from the following argument: Assume Z lies along one of the sides, say UV. Then the locus of equation (9) passes through Z and is symmetric about UV. If u and v are unequal, then the tangent to the curve at Z is perpendicular to UV. Since WZ must be perpendicular to the tangent, then U, V, and W must be collinear. Thus, for U, V, and W not collinear and $u \neq v$, the solution cannot be along UV. If, however, $v > u$ and the constant in equation (9) is $u\overline{UV}$, then the locus such that $ur_U + vr_V = u\overline{UV}$ is just the point V and $\overline{WV}$ is the unique (and hence, minimum) distance from W to this locus. Thus we have shown that if u, v, and w are all different, the optimum solution lies either at a vertex or in the interior of the triangle.

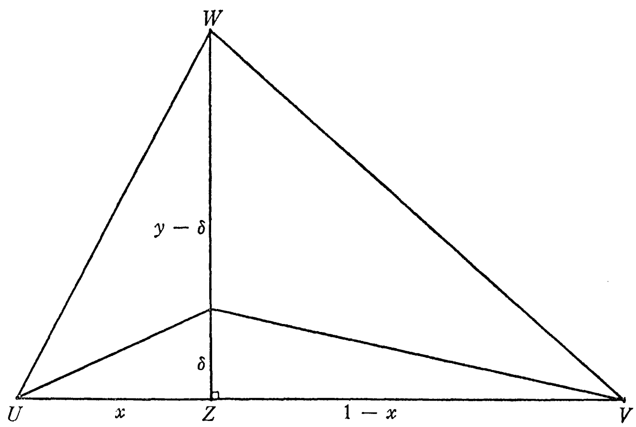

FIG. 3. u and v equal.

If u and v are equal, then the locus of equation (9) could be the straight line UV and the above argument cannot be used. Instead, the proof can be obtained from Figure 3. If the solution lies along UV, it is at the intersection of UV and the perpendicular from W. Let the length of this perpendicular be y and let the distance $\overline{UV} = 1$; the intersection occurs a distance x from U. It can be shown that there exists at least one point along WZ which gives a smaller total cost; this point is located a distance δ above UV. The cost is

$$u\sqrt{(x^2 + \delta^2)} + u\sqrt{((1 - x)^2 + \delta^2)} + w(y - \delta).$$

The value of δ which minimizes the cost is obtained by differentiating with respect to δ and equating to zero, which yields

$$u\delta\left[\frac{1}{\sqrt{(x^2+\delta^2)}}+\frac{1}{\sqrt{((1-x)^2+\delta^2)}}\right]-w=0.$$

Clearly, $\delta=0$ cannot be a solution, but δ arbitrarily small (but positive) makes the derivative negative. Thus, increasing δ from zero by this small amount decreases the cost; hence, the optimum cannot occur along UV.

Thus far, we have shown that the solution must lie inside the triangle or at a vertex, and if it is at a vertex, it is the vertex receiving the maximum shipment, assuming that this maximum is unique. In the case of all factories receiving the same shipment ($u=v=w$) then $a=b=c=30°$ and $\alpha=\beta=\gamma=120°$. If one of the angles of the triangle is greater than 120°, the solution will lie at the vertex of that angle. In the case $u=v>w$ and the solution does not fall in the triangle, it is at U or V, whichever has the largest angle. These solutions can be verified by determining the total cost of shipping from each vertex and choosing the one which yields the minimum cost.

NEW EXPERIMENTAL RESULTS ON THE GOLDBACH CONJECTURE

M. L. STEIN AND P. R. STEIN, Los Alamos Scientific Laboratory

1. Introduction. Goldbach's famous conjecture—that every even number can be represented in at least one way as the sum of two primes—is now 222 years old and remains unproved despite overwhelming evidence for its truth. The most important advance in this century was provided by the work of I. Vinogradov [1], who showed that all odd numbers greater than some unspecified (large) integer can be expressed as the sum of three primes. If Goldbach's conjecture were proved, Vinogradov's result would follow trivially, but nothing can be proved about the former by assuming the latter.

In view of the failure of analytic methods to decide the problem, we thought it might be helpful to undertake a detailed experimental study of the so-called Goldbach curve—that is, the number of distinct solutions ν_{2n} of the equation $2n=p_i+p_j$—in the hope of turning up some new facts which might suggest a fresh approach to the central problem. Whether or not we have been successful in this remains to be seen. In any event, our work has led to three new conjectures and to a heuristic understanding of the actual Goldbach curve, at least in the finite interval $2n<150{,}000$. These matters are discussed in the next four sections.

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The analogue of Goldbach's conjecture for primes—hereafter referred to as

G(P)—can also be made for other "sieve" numbers. In particular, it appears to hold for the "lucky numbers" first defined by S. Ulam [2]. Further, the three conjectures described below can also be plausibly put forward in the "lucky" case. A brief description of the results for lucky numbers is given in Section 6.

2. The first conjecture. The raw data for this investigation is a table of the number of "Goldbach decompositions" ν_{2n} for all even numbers in the range $2n < 150{,}000$. (Since this article was written this table and the corresponding one for lucky numbers have been published as a Los Alamos report LA 3106, Vol. I and II. The range of these tables is $2n < 200{,}000$.) This table was calculated on the Laboratory's MANIAC II electronic computer. It has been carefully checked, both by repetition and by numerous independent "spot-checks" for selected values of $2n$ [4].

The first conjecture to be based on this data is as follows.

CONJECTURE I (P). *For every integer $k > 0$ there exists at least one even number $2n$ such that $\nu_{2n} = k$.*

This conjecture—which was made in the early stages of the calculation—now appears quite solid. It is true for all $k \leqq 1911$. The first gap is at $k = 1912$; the next few gaps occur for $k = 1942$, 2078, 2113, 2140. (After this, the gaps become fairly frequent.) On the basis of experience, we should expect that at least one solution of $\nu_{2n} = 1912$ would be found by extending our table up to $2n < 160{,}000$. The largest value of ν_{2n} found in our range is 2969, which occurs for $2n = 143{,}220$.

It is not at all obvious that this conjecture is dependent on the truth of G(P); its precise status with regard to the latter remains to be investigated. Experimentally, the number of solutions of the equation $\nu_{2n} = k$ for given k is quite respectable. To date we have not looked at this new curve in detail.

3. The second conjecture. The next conjecture suggested by our data may be stated as follows.

CONJECTURE II(P). *For every integer $k > 8$, the smallest solution $2n$ of the equation $\nu_{2n} = k$ is such that $2n \equiv 0 \pmod 6$.*

In fact, this holds for all $k > 4$ for which any solution has been found so far, with the exception of $k = 8$. (The first value of $2n$ for which $\nu_{2n} = 8$ is 140.) The conjecture is rendered somewhat more plausible than it appears at first sight by noting that the Goldbach curve "almost always" has a local maximum at points $2n \equiv 0 \pmod 6$. This in turn is to be expected in view of the fact that there are—very roughly—twice as many decompositions $6m = p_i + p_j$ as there are $6m \pm 2 = p_i + p_j$, since in the first case primes of both parities $6j \pm 1$ are available. Actually, all even numbers $36 < 2n < 150{,}000$ which are divisible by 6 have more Goldbach decompositions than their immediate neighbors, with two exceptions: there is one "contact" ($\nu_{1542} = \nu_{1540} = 46$) and one "crossing" ($\nu_{80080} = 1006$, $\nu_{80082} = 1005$).

4. The structure of the Goldbach curve. The detailed structure of the Goldbach curve is very irregular; the magnitude and location of the various maxima and minima would seem to be quite unpredictable. Nevertheless, it turns out that a refinement of the crude argument in the previous section (concerning the local maxima at points $2n \equiv 0 \pmod 6$) yields a prescription which enables one to predict the curve in remarkable detail. Using this prescription, we have been able to fit the entire curve in the range $30{,}000 \leqq 2n < 150{,}000$ with an overall (absolute) error of 2.63% and a maximum error of 13.74% (which occurs at the point $2n = 33{,}038$).

Before describing the formula, we remark that it is clear why an even number whose factorization contains a relatively large number of distinct primes should have a correspondingly large number of Goldbach decompositions. This is simply due to the fact that there are fewer composites of the form $2n-p$, since all composites sharing a prime factor of $2n$ are excluded. Consequently, there is a correspondingly greater chance that $2n-p$ is a prime. It is therefore evident that the actual factor structure of $2n$ is important in determining the size of ν_{2n}. Another relevant consideration is the "parity" (or residue class) of $2n$ modulo some conveniently chosen integer. (These considerations are, of course, not independent, but it is convenient to consider them so for our purposes.)

The existence of certain local maxima is "explained" by considering the even numbers modulo 6. We can get more detailed results by using, say, $30 = 2\cdot 3\cdot 5$ or even $210 = 2\cdot 3\cdot 5\cdot 7$, etc. This leads us to define the following quantities.

Let Π_k be the product of the first k primes where we take 2 as the first prime. Let n_s be the number of solutions of

$$s \equiv r_i + r_j \pmod{\Pi_k} \tag{1}$$

where r_i, r_j, belong to the set of $\phi(\Pi_k)$ integers prime to Π_k, and s is an even number, $0 \leqq s < \Pi_k$. For our purposes, we take $r_i + r_j$ and $r_j + r_i$ as distinct unless $r_i = r_j$.

Now the possible values of s fall into 2^{k-1} residue "types" according to which primes of the set $\{2, 3, 5, 7, \cdots, p_k\}$ appear as distinct factors of s. Thus each s is of the form $2^{\alpha_1}3^{\alpha_2}5^{\alpha_3} \cdots p_k^{\alpha_k}$. R, where R does not contain any of the first k primes as a factor. A residue type is completely defined by specifying which of the α_i are nonzero, but is independent of their actual values.

It is easy to see that n_s is the same for all s belonging to a given residue type. We may label the residue types (the order is arbitrary) by the index i: $1 \leqq i \leqq 2^{k-1}$. For all s belonging to the residue type i we write:

$$n_s = g_i^{(k)}. \tag{2}$$

For example, if $k=4$, the values of $g_i^{(k)}$ are as follows (here we specify the residue type by giving the actual primes that appear):

TABLE I

Type	$g_i^{(k)}$
(2)	15
(2, 3)	30
(2, 5)	20
(2, 7)	14
(2, 3, 5)	40
(2, 3, 7)	36
(2, 5, 7)	24
(2, 3, 5, 7)	48

It is easy to show that the $g_i^{(k)}$ may be calculated by recursion on k:

a. If i is a residue type for Π_k which does not contain the prime p_k, then

$$g_i^{(k)} = (p_k - 2)g_i^{(k-1)}. \tag{3}$$

b. If i does contain the prime p_k, then

$$g_i^{(k)} = (p_k - 1)g_{i^*}^{(k-1)}, \tag{4}$$

where i^* is the residue type which results on omitting the prime p_k.

As defined above, the $g_i^{(k)}$ are the numbers of *different integers* prime to Π_k (rather than the number of *solutions*) which appear as solutions of equation (1). Thus, for $k=3$:

$s = 0$:

$$\left.\begin{aligned} 1 + 29 &\equiv 0 \pmod{30} \\ 7 + 23 &\equiv 0 \pmod{30} \\ 11 + 19 &\equiv 0 \pmod{30} \\ 13 + 17 &\equiv 0 \pmod{30} \end{aligned}\right\} g = 8$$

$s = 6$:

$$\left.\begin{aligned} 7 + 29 &\equiv 6 \pmod{30} \\ 13 + 23 &\equiv 6 \pmod{30} \\ 17 + 19 &\equiv 6 \pmod{30} \end{aligned}\right\} g = 6$$

$s = 2$:

$$\left.\begin{aligned} 1 + 1 &\equiv 2 \pmod{30} \\ 13 + 19 &\equiv 2 \pmod{30} \end{aligned}\right\} g = 3$$

$s = 10$:

$$\left.\begin{aligned} 11 + 29 &\equiv 10 \pmod{30} \\ 17 + 23 &\equiv 10 \pmod{30} \end{aligned}\right\} g = 4$$

From these values of the $g_i^{(.3)}$, Table I is easily calculated via equation (3) and (4).

We are now ready to give our prescription for predicting ν_{2n}. Let $2n$ have known factorization $2^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots$, and let it be of residue type i (mod Π_k). Let $P_a^{(k)}$ be the number of primes $\leqq n$, excluding $1, 3, 5, \cdots, p_k$, and let P_b be the number of primes between n and $2n$. Let $D^{(k)}$ be the number of integers $<n$ which are prime to $2, 3, 5, \cdots, p_k$ and to any additional prime factors appearing in the factorization of $2n$. The "expected" number $E_{2n}(\nu)$ of Goldbach decompositions of $2n$ is then given by the expression:

$$E_{2n}(\nu) = \frac{g_i^{(k)}}{\phi(\Pi_k)} \frac{P_a^{(k)}}{D^{(k)}} P_b. \tag{5}$$

Here the collection of factors multiplying P_b is essentially an approximate expression for the probability that $2n-p$ is a prime, p being a prime of appropriate parity from the upper half of the interval. In this formula, $D^{(k)}$ is easily calculated by the well-known "Inclusion-Exclusion" algorithm once the factorization of $2n$ is known. The residue type (mod Π_k) is also determined in an obvious manner. The calculation of the $g_i^{(k)}$ has already been described. $P_a^{(k)}$, P_b may, of course, be obtained from a table of primes (or, equally well, from some analytic approximation, with sufficient accuracy for this purpose). Thus there are really no empirical constants in the formula.

We have applied this formula to every even number in the range $30{,}000 \leqq 2n < 150{,}000$ for $k=5$, i.e., for $\Pi_k = 2\cdot 3\cdot 5\cdot 7\cdot 11 = 2310$. For each case we have calculated the percent error:

$$\epsilon = 100(E_{2n}(\nu) - \nu_{2n})/\nu_{2n}. \tag{6}$$

Note that, as defined, $E_{2n}(\nu)$ is not generally an integer; we have not bothered to round it off.

ϵ is positive in the great majority of cases, but for a few even numbers our formula actually gives an underestimate; in these cases the absolute error is small. In Table II we list the number of cases for which the absolute error lies between specified limits.

TABLE II

Absolute % Error	Number of Cases	Absolute % Error	Number of Cases
$\lvert\epsilon\rvert < .05$	531	$5.00 \leqq \lvert\epsilon\rvert < 6.00$	2902
$.05 \leqq \lvert\epsilon\rvert < 1.00$	9875	$6.00 \leqq \lvert\epsilon\rvert < 7.00$	1295
$1.00 \leqq \lvert\epsilon\rvert < 2.00$	13161	$7.00 \leqq \lvert\epsilon\rvert < 8.00$	600
$2.00 \leqq \lvert\epsilon\rvert < 3.00$	14179	$8.00 \leqq \lvert\epsilon\rvert < 9.00$	235
$3.00 \leqq \lvert\epsilon\rvert < 4.00$	10949	$9.00 \leqq \lvert\epsilon\rvert < 10.00$	104
$4.00 \leqq \lvert\epsilon\rvert < 5.00$	6098	$\lvert\epsilon\rvert > 10.00$	71

As stated in the Introduction, the maximum error is 13.74%, and occurs for $2n = 33{,}038$; here $\nu = 224$, $E = 254.78$.

Finally, we have calculated ν_{2n} and $E_{2n}(\nu)$ for three large values well outside the range of our table ($k=5$):

TABLE III

$2n$	ν_{2n}	$E_{2n}(\nu)$	ϵ
1000000	5402	5533.46	2.43
1000002	8200	8300.07	1.22
1000004	4161	4277.85	2.81

5. Minimal primes: the third conjecture. In the light of the experimental evidence, G(P) appears to be a rather modest conjecture. This is illustrated by Table IV, which gives values of N^* and ν^* such that, if $2n > N^*$, then $\nu_{2n} > \nu^*$.

TABLE IV

N^*	ν^*	N^*	ν^*
4688	50	63962	400
11672	100	75188	450
19246	150	85616	500
27908	200	95276	550
36242	250	105368	600
45998	300	116618	650
55446	350	126878	700

In other words, all even numbers greater than 85616 have more than 500 Goldbach decompositions, etc. Table IV is, of course, based on calculations for the range $2n < 150{,}000$, but the general trend of the curve makes it unlikely that the table will be found incorrect.

All in all, it would appear that the primes form a sequence that is much denser than is necessary for G(P) to hold. Accordingly, we thought it would be of interest to construct a subsequence of the primes for which G(P) holds by construction and to study its density. Naturally, any such sequence will terminate if G(P) is false and may terminate even if G(P) is true.

The sequence we have worked with may be defined recursively in the following manner. Let the first m terms of our sequence be $\{p_1, p_2, p_3, \cdots, p_m\} \equiv (M)$. Form all even numbers of the form $2n = p_i + p_j$, where $p_i, p_j \in (M)$. Let $2n^*$ be the first even number not so expressible. We then choose the largest prime $p < 2n^*$ such that $p + p_j = 2n^*$, $p_j \in (M)$. We then set $p_{m+1} = p$ and continue.

The first few terms of the sequence are: 1, 3, 7, 11, 13, 17, 31, 29, 47, 41, 53, 67, 83, 103, 109, 127, 139, 137, 157, 181, $\cdots$. Note that the algorithm does not produce these primes—we shall call them "minimal primes"—in strictly increasing order; this "backtracking" appears to be an inescapable feature of the prescription which persists as the sequence is carried to higher values.

The minimal prime sequence has been successfully extended up to $2n = 1{,}000{,}000$. In this range there are exactly 3,000 terms, the last one being 999,043; this is the 78,437th prime in the natural sequence. From a log-log plot it appears that

$$m \sim \Pi^r(p_m) \quad \text{with} \quad r \cong 0.6 \text{ or } 0.59, \tag{7}$$

where p_m is the mth minimal prime (in the natural order) and $\pi(x)$ is, as usual, the number of primes $\leq x$. On the basis of these results, we are led to make

CONJECTURE III (P). *The minimal prime sequence is infinite.*

This is a much stronger conjecture than G(P). If it should be proven false, that is, if the sequence should terminate, we can always insert a new prime pair and continue (this will be possible if G(P) holds). In our opinion, however, the conjecture as it stands is not implausible.

It turns out that, in a certain sense, the algorithm is quite efficient. As each new minimal prime is determined, one of the earlier values appears as a "complement" $\bar{p}$, i.e., $\bar{p}=2n^*-p_{m+1}$. Up to $2n=1{,}000{,}000$, only 47 minimal primes are used as complements; this shows that the "backtracking," while persistent, is never extreme. (223 of the 3000 minimal primes are generated "out of order".) In Table V we list the minimal primes which appear as complements along with the number of times each prime is so used. In this table, the primes are listed in their "algorithmic" rather than their natural order.

TABLE V

m	p_m	frequency	m	p_m	frequency	m	p_m	frequency
1	1	332	17	127	44	33	359	9
2	3	606	19	139	34	34	379	1
3	7	252	18	137	25	35	401	3
4	11	329	20	157	19	36	421	1
5	13	134	21	181	20	37	457	1
6	17	256	22	193	5	38	461	5
8	31	97	24	199	9	40	509	3
7	29	172	23	197	21	41	521	6
10	47	86	25	229	3	43	569	2
9	41	107	26	239	12	45	617	1
11	53	73	27	251	20	47	653	1
12	67	88	28	271	6	50	709	1
13	83	57	29	307	5	53	773	3
14	89	43	30	313	1	51	743	1
15	103	43	31	317	12	71	1319	1
16	109	48	32	349	2			

The regular alternation of frequency in the first part of the table is striking, but we hesitate to suggest at this stage that it is significant.

6. Lucky numbers. The experiments described above may also be carried out for the case where, instead of prime numbers, we use another sequence of "sieve" numbers called "lucky numbers" [2]. We recall that, according to the sieve of Eratosthenes, the primes are determined successively by, at the jth step, marking every p_jth number counting from the jth prime p_j. The first number greater than p_j which remains unmarked is the $j+1$st prime. In counting from the p_jth prime, all numbers remain in the list regardless of whether or not they have previously been marked. In the lucky number sieve, on the other hand, a number is removed as soon as it has been marked. Thus, having removed all the even numbers from the list of integers, we find that the first number >1 which remains is 3. We then strike out every third *remaining* number, counting from the beginning (this is the usual convention). Thus every number of the form $6k-1$ is eliminated. The first remaining number >3 is 7, so we strike out every 7th number counting from the beginning, etc. The elements of the resulting infinite sequence are called lucky numbers. It has been shown [3] that these numbers have, to lowest order, the same asymptotic density as the primes.

In contrast to the prime case, there is a convenient algorithm for calculating the kth lucky number directly in terms of the luckies $\leqq k$. Since this may not be well known, we give it here.

Let l_{m-1} be the greatest lucky $\leqq k$. Set $R_m = k$. Then define:

(8a) $$j = \left[\frac{R_m}{l_{m-1} - 1}\right] \quad \text{if } l_{m-1} - 1 \text{ does not divide } R_m$$

(8b) $$j = \left[\frac{R_m}{l_{m-1} - 1}\right] - 1 \quad \text{if } l_{m-1} - 1 \text{ divides } R_m$$

(9) $$R_{m-1} = R_m + j.$$

We continue the process down to $m = 2$. Then, finally:

(10) $$l_k = 2R_2 - 1.$$

This algorithm—which follows directly from the definition of the lucky sieve—is extremely useful in checking the table of luckies; the latter, of course, is best calculated by carrying out the sieve process as originally defined. Using MANIAC II, we have computed (and checked) a table of the first 36,655 luckies.

In the original paper [2] it was more or less implied that the lucky number analogue of Goldbach's conjecture—call it G(L)—holds, i.e., that every even number has at least one decomposition into two luckies. In fact, the authors of [2] verified this property for the range $2n \leqq 100{,}000$. In parallel with our work on the Goldbach curve for primes, we also calculated a table of the number λ_{2n} of solutions of the equation $2n = l_i + l_j$, where l_i, l_j are luckies. This was done over the same range, $2n < 150{,}000$. The resulting curve is, in its gross aspects, quite similar to the usual Goldbach curve. The local maxima, however, now occur at values $2n = 6k - 2$. This is to be expected, since there are no luckies of the form $6k-1$. (There are no "crossings" (in the sense of Section 2) and one "contact": $\lambda_{190} = \lambda_{192} = 7$.)

The conjectures analogous to those made above for primes can equally well be made for this case:

CONJECTURE I (L). *For every integer $k > 0$ there exists at least one even number $2n$ such that $\lambda_{2n} = k$. This has been verified for all $k \leqq 1769$.*

CONJECTURE II (L). *For every integer $k > 1$, the smallest solution $2n$ of $\lambda_{2n} = k$ is such that $2n \equiv -2 \pmod 6$. This holds without exception for all k for which any solution has been found to date.*

CONJECTURE III (L). *The minimal lucky sequence is infinite. The sequence has been carried successfully up to* $2n=350{,}000$ *(1673 terms). So far as we can tell, the same asymptotic law appears to hold, i.e., the number of minimal luckies seems to go as roughly the 0.6th power of the number of luckies.*

So far we have not been able to devise a formula which satisfactorily predicts the number of lucky decompositions of a given even number. This is perhaps due to the fact that there is no simple analogue of the unique factorization property (and hence nothing like the Euler ϕ function). By working with "pseudo-luckies," i.e., those numbers remaining after a fixed number of steps of the sieve process, we can calculate ratios analogous to the $g_i^{(k)}/\phi(\Pi_k)$ of Section 4; these do, in fact, enable us to predict the *relative* magnitudes of successive λ_{2n} to 10–15%.

It is clear, however, that a more refined technique is necessary in this case.

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4. M. L. Stein and P. R. Stein, Tables of the number of binary decompositions of all even numbers $0<2n<200{,}000$ into prime numbers and lucky numbers, Los Alamos Rep. LA-3106, vol. I and II, available from the Office of Tech. Serv., U. S. Dept. of Commerce, Washington 25, D. C.

THE ICOSAHEDRAL GROUP OF LINEAR TRANSFORMATIONS IN THE PLANE

G. H. LUNDBERG, Vanderbilt University

Here is shown a method of computing several linear transformations in the plane as applied to the rotation of the icosahedron and the dodecahedron. In addition, there is shown a method of transforming the 2nd, 3rd, and 5th order groups.

The rotation of the icosahedron and the dodecahedron. The group of movements which carry the regular icosahedron into itself are sixty in number. These movements also carry the regular dodecahedron into itself because of the duality existing between the solids. The twenty faces of the icosahedron correspond to the like number of vertices of the dodecahedron, and the twelve faces of the latter correspond to the same number of vertices of the former. Both have thirty edges. Thus, it is seen that the figures furnish the same set of axes. The six axes of the fifth order groups are formed in the icosahedron by joining opposite pairs of the twelve vertices and, in the dodecahedron, by connecting opposite pairs of midpoints of the twelve pentagonal faces. Furthermore, the ten axes of rota-

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The transformations in the complex plane which correspond to each rotation are derived in this article by considering the icosahedron. The results obtained are applicable also to the movements of the dodecahedron.

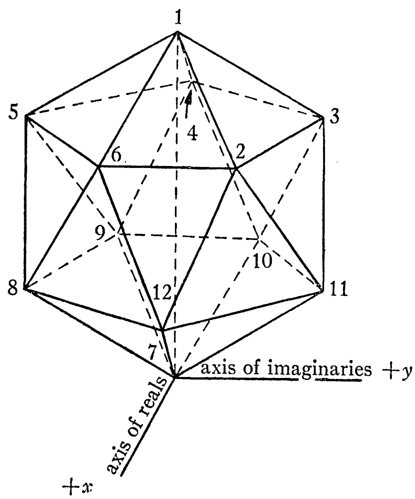

FIG. 1.

Computation of certain transformations. In order to find the transformations in the complex plane which correspond to the rotations which carry the icosahedron into itself, it is necessary to derive first certain auxiliary transformations. Fig. 1 shows the icosahedron inscribed in a sphere, which is tangent to the complex plane at the origin.

Fig. 2 shows the section of the solid in the plane determined by the axis of reals and the axis of the sphere perpendicular to the plane at the origin, when viewed from the positive side of the axis of imaginaries.

From this section, it can be observed that the trigonometric functions of the acute angle $A/2$ of the right triangle $O'OL$ are $\cos A/2=(r-h)/r$ and $\sin A/2 =a/2r$. Since the angles $A/2$ and B are complementary, $\cos B=\sin A/2=a/2r$. By the Pythagorean Theorem, $(r-h)^2+a^2/4=r^2$, and by the Law of Cosines, $3(a^2/4)=(r-h)^2+r^2-2r(r-h)\cos B$. When the value $a/2r$ is substituted for $\cos B$ in the foregoing equation, and the equation immediately preceding it is subtracted from it, the result is $2(r-h)^2-a(r-h)-a^2/2=0$. If the negative root is disregarded,

$$r - h = \frac{a(1 + \sqrt{5})}{4},$$

whence

$$(r-h)^2 = \frac{a^2(3+\sqrt{5})}{8}.$$

Substituting the value for $(r-h)^2$ in the equation, $r^2=(r-h)^2+a^2/4$, gives

$$r^2 = \frac{a^2(5+\sqrt{5})}{8}.$$

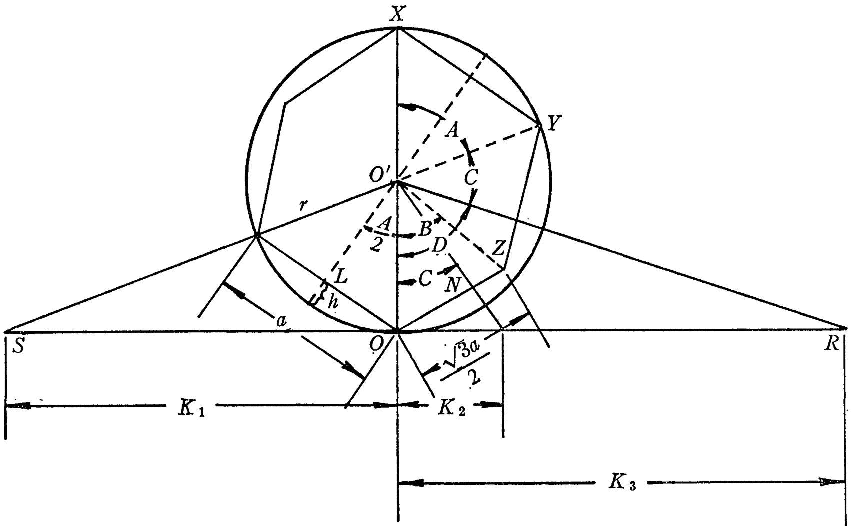

FIG. 2.

The cosine of any angle, in terms of the cosine of its half, is $\cos A = 2\cos^2 A/2 - 1$. The substitutions of first the value of $\cos A/2$ and then the values for $(r-h)^2$ and r^2 give $\cos A = \sqrt{5}/5$. Thus, $\tan A = 2$. In the right triangle $O'OS$, $\tan A = K_1/r$. If the radius of the sphere is $\frac{1}{2}$, then $K_1 = 1$, which represents a point in the complex plane on the axis of reals.

The general transformation in the complex plane for any rotation of the sphere about an axis parallel to the axis of imaginaries is

$$z' = \frac{az+b}{-bz+a}.$$

If K_1 is carried into the origin by a counterclockwise rotation,

$$\frac{a+b}{-b+a} = 0,$$

then $a=-b$. The transformation in the complex plane corresponding to a rotation of the sphere with an axis parallel to the axis of imaginaries, which brings vertex Y to the position occupied by X, is

$$z' = \frac{z-1}{z+1}\,.$$

The inverse transformation which sends the icosahedron back to its first position is

$$z' = \frac{z+1}{-z+1}\,.$$

Since the line OZ is the altitude on one side of an equilateral triangle with side a,

$$OZ = \frac{\sqrt{3}\,a}{2}\,.$$

Then $ON=\sqrt{3}a/3$ for the N lies at the centroid of a face of the icosahedron. In triangle $O'NO$, the Pythagorean relationship gives

$$\overline{O'N}^2 = r^2 - \left(\frac{\sqrt{3}a}{3}\right)^2.$$

On substituting for r^2 we obtain

$$\overline{O'N}^2 = \left(\frac{7+3\sqrt{5}}{24}\right)a^2,$$

and

$$O'N = \frac{a(3+\sqrt{5})}{4\sqrt{3}}\,.$$

Substituting this value of $O'N$ in the trigonometric form, $\tan C=\sqrt{3}a/3O'N$, derived from the right triangle $O'NO$, gives $\tan C=3-\sqrt{5}$. From the right triangle $OO'T$, $\tan C=-K_2/r$; then on substitution of values for $\tan C$ and r, the result is $K_2=-\frac{1}{2}(3-\sqrt{5})$, which represents a point in the complex plane on the axis of reals.

Using the same transformation,

$$z' = \frac{az+b}{-bz+a},$$

to carry the K_2 into O in a clockwise rotation about the axis, we obtain the relation between the constants $b=\frac{1}{2}a(3-\sqrt{5})$, making the specific transformation for this rotation

$$z' = \frac{2z+3-\sqrt{5}}{(-3+\sqrt{5})z+2},$$

the inverse of which is

$$z' = \frac{2z - 3 + \sqrt{5}}{(3 - \sqrt{5})z + 2}.$$

Since the angles $(A+C)$ and D are supplementary, we have that

$$\tan D = -\tan(A + C) = \frac{\tan A + \tan C}{\tan A \tan C - 1}.$$

On substitution of the values for $\tan A$ and $\tan C$ and simplifying, we obtain $\tan D = 3+\sqrt{5}$. In right triangle $O'OR$, $\tan D = -K_3/r$, whence $K_3 = -\frac{1}{2}(3+\sqrt{5})$.

Then if K_3 is carried into the origin by a clockwise rotation, the relation between the constants in the general transformation

$$z' = \frac{az + b}{-bz + a}, \quad \text{is} \quad b = \frac{1}{2}a(3 + \sqrt{5});$$

and the specific transformation for this rotation becomes

$$z' = \frac{2z + 3 + \sqrt{5}}{-(3 + \sqrt{5})z + 2},$$

whose inverse is

$$z' = \frac{2z - 3 - \sqrt{5}}{(3 + \sqrt{5})z + 2}.$$

Transformations of fifth order groups. In the transformations of the icosahedron given in this article, each of the powers of R indicates one of the fifth roots of unity. In this section, the method of deriving six groups of order five, consisting of rotations about axes determined by the pairs of vertices (1, 7), (4, 12), (5, 11), (2, 9), (3, 8), and (6, 10) of Fig. 1, is given.

The transformations in the complex plane which carry the icosahedron into itself when the poles of the axis are 1 and 7 will consist of

$$z' = Rz, \qquad z' = R^2z, \qquad z' = R^3z, \qquad z' = R^4z, \quad \text{and} \quad z' = z.$$

Here, each rotation is through one of the fifth roots of unity.

In order to find the first transformation of the fifth order group when the poles of the axis are at 4 and 12, the solid is first carried through a counterclockwise rotation about the axis parallel to the axis of imaginaries by means of the first auxiliary transformation,

$$z' = \frac{z - 1}{z + 1},$$

then counterclockwise about the perpendicular axis through the first fifth root of unity, giving

$$z' = \frac{Rz - 1}{Rz + 1},$$

and, finally, clockwise about the axis parallel to the axis of imaginaries by the inverse of the auxiliary transformation, resulting in

$$z' = \frac{z(R + 1) + R - 1}{z(R - 1) + R + 1}.$$

This result corresponds to the counterclockwise rotation about the axis determined by vertices 4 and 12 through an angle of 72°.

Repeating this transformation gives

$$z' = \frac{(R^2 + 1)z + R^2 - 1}{(R^2 - 1)z + R^2 + 1},$$

which corresponds to a counterclockwise rotation of 144° about the same axis. This result can also be obtained directly by using the following transformations in the order given:

$$z' = \frac{z - 1}{z + 1}, \qquad z' = R^2 z, \quad \text{and} \quad z' = \frac{z + 1}{-z + 1}.$$

The remaining three transformations of this fifth order group can likewise be found by three more successive repetitions or directly by using, in turn, the third, fourth, and fifth roots of unity between the auxiliary transformation and its inverse. These are given below in the order of their rotations of 216°, 288°, and 360°:

$$z' = \frac{(R^3 + 1)z + R^3 - 1}{(R^3 - 1)z + R^3 + 1}, \qquad z' = \frac{(R^4 + 1)z + R^4 - 1}{(R^4 - 1)z + R^4 + 1},$$

and $z' = z$.

Each set of poles (5, 11), (2, 9), (3, 8), and (6, 10) is, in turn, rotated about the perpendicular axis through one of the fifth roots of unity, so that their new positions in the plane are determined by the axis of reals and the perpendicular at the origin. Then, the solid in each case is rotated back about the perpendicular axis through the inverse angle, after the same rotations are performed as were performed with the solid when the poles were 4 and 12. Thus, all the fifth order transformations of each set of poles can be found. An illustration follows: if the poles are 5 and 11, the transformation for a counterclockwise rotation of 72° about their axis is found by using the following transformations in the order given:

$$z' = R^4 z, \qquad z' = \frac{z - 1}{z + 1}, \qquad z' = Rz, \qquad z' = \frac{z + 1}{-z + 1}$$

and $z'=Rz$. This results in the transformation,

$$z' = \frac{(1+R)z+1-R^4}{(R^2-R)z+1+R}.$$

Similarly other members of this set are found to be

$$z' = \frac{(R^2+1)z+R-R^4}{(R^3-R)z+R^2+1}, \qquad z' = \frac{(R^3+1)z+R^2-R^4}{(R^4-R)z+R^3+1},$$

$$z' = \frac{(R^4+1)z+R^3-R^4}{(1-R)z+R^4+1}, \qquad z' = z.$$

The transformations of the three fifth order groups having as their respective poles of rotation (2, 9), (3, 8), and (6, 10) are, respectively,

$$z' = \frac{(R+1)z+R^3-R^2}{(R^4-R^3)z+R+1}, \qquad z' = \frac{(R^2+1)z+R^4-R^2}{(1-R^3)z+R^2+1},$$

$$z' = \frac{(R^3+1)z+1-R^2}{(R-R^3)z+R^3+1}, \qquad z' = \frac{(R^4+1)z+R-R^2}{(R^2-R^3)z+R^4+1}, \quad \text{and} \quad z' = z;$$

$$z' = \frac{(R+1)z+R^2-R}{(1-R^4)z+R+1}, \qquad z' = \frac{(R^2+1)z+R^3-R}{(R-R^4)z+R^2+1},$$

$$z' = \frac{(R^3+1)z+R^4-R}{(R^2-R^4)z+R^3+1}, \qquad z' = \frac{(R^4+1)z+1-R}{(R^3-R^4)z+R^4+1}, \quad \text{and} \quad z' = z;$$

and

$$z' = \frac{(R+1)z+R^4-R^3}{(R^3-R^2)z+R+1}, \qquad z' = \frac{(R^2+1)z+1-R^3}{(R^4-R^2)z+R^2+1},$$

$$z' = \frac{(R^3+1)z+R-R^3}{(1-R^2)z+R^3+1}, \qquad z' = \frac{(R^4+1)z+R^2-R^3}{(R-R^2)z+R^4+1}, \quad \text{and} \quad z' = z.$$

Transformations of the third order groups. If the first position of the icosahedron is assumed to be as it is in Fig. 1, it is seen that the clockwise rotation about the axis parallel to the axis of imaginaries represented by the second auxiliary transformation,

$$z' = \frac{2z+3-\sqrt{5}}{(-3+\sqrt{5})z+2},$$

brings two opposite faces, represented by the two pairs of triple sets of vertices (1, 2, 6) and (7, 9, 10) to a new position where both are parallel to the plane. The axis of rotation for these pairs of opposite faces will now be perpendicular to the plane at the origin. Next, if the solid is rotated in a counterclockwise direction about the perpendicular axis through the first cube root of unity represented by ω, the product transformation will be

$$z' = \frac{2\omega z + 3 - \sqrt{5}}{(-3\omega + \sqrt{5}\omega)z + 2};$$

and, finally, when the solid is moved in the clockwise direction about the axis parallel to the axis of imaginaries by the inverse of the auxiliary transformation given above, the resulting transformation is

$$z' = \frac{(-2\omega - 7 + 3\sqrt{5})z + (\omega - 1)(3 - \sqrt{5})}{(\omega - 1)(3 - \sqrt{5})z + (-2 - 7\omega + 3\sqrt{5}\omega)}.$$

This is the first member of the third order set which has its upper face (1, 2, 6) in the position shown in Fig. 1. The movements made have precisely the same effect as rotating the solid counterclockwise through an angle of 120° about the axis of the faces in the position they were originally.

In a similar manner, using the following transformations in the given order,

$$z' = \frac{2z + 3 - \sqrt{5}}{(-3 + \sqrt{5})z + 2}, \qquad z' = \omega^2 z, \quad \text{and} \quad z' = \frac{2z - 3 + \sqrt{5}}{(3 - \sqrt{5})z + 2},$$

the second member of this third order set is

$$z' = \frac{(-2\omega^2 - 7 + 3\sqrt{5})z + (\omega^2 - 1)(3 - \sqrt{5})}{(\omega^2 - 1)(3 - \sqrt{5})z + (-2 - 7\omega^2 + 3\sqrt{5}\omega^2)},$$

and, similarly, the third member becomes the identity, $z' = z$. The second and third members may also be found by one and by two repetitions, respectively, of the first member of this third order group.

The transformations of the four third order groups corresponding to the rotations about axes passing through each pair of the remaining opposite faces arranged around the vertices 1 and 7 may be found in the following manner: the icosahedron is first rotated about the axis perpendicular to the plane through one of the fifth roots of unity. Next, one set of movements of the third order group just derived is made. Then, finally, the solid is turned back clockwise through the inverse of the first rotation.

For example, the first member of the third order group of transformations corresponding to the counterclockwise rotation of 120° about the axis which passes through the opposite triangles determined by the sets of vertices (1, 5, 6) and (7, 11, 12), is found by using the following transformations in the order given:

$$z' = Rz, \qquad z' = \frac{2z + 3 - \sqrt{5}}{(-3 + \sqrt{5})z + 2}, \qquad z' = \omega z, \qquad z' = \frac{-2z + 3 - \sqrt{5}}{(-3 + \sqrt{5})z - 2},$$

and $z' = R^4 z$. The resulting transformation is

$$z' = \frac{(-2\omega - 7 + 3\sqrt{5})z + (\omega - 1)(3R - \sqrt{5}R)}{(\omega - 1)(3R^4 - \sqrt{5}R^4)z + (-2 - 7\omega + 3\sqrt{5}\omega)}.$$

Similarly, when the middle transformation is $z'=\omega^2 z$, instead of $z'=\omega z$, the second member of this set becomes

$$z' = \frac{(-2\omega^2 - 7 + 3\sqrt{5})z + (\omega^2 - 1)(3R - \sqrt{5}R)}{(\omega^2 - 1)(3R^4 - \sqrt{5}R^4)z + (-2 - 7\omega^2 + 3\sqrt{5}\omega^2)} .$$

Finally when the middle transformation is $z'=\omega^3 z$ or the identity, the result will be the identity, $z'=z$. The second two members may be derived also by successive repetitions of the first.

Analogously, the transformations of the three remaining third order groups whose axes pass through the upper faces of the icosahedron are:

$$z' = \frac{(-2\omega - 7 + 3\sqrt{5})z + (\omega - 1)(3R^2 - 2\sqrt{5}R^2)}{(\omega - 1)(3R^3 - \sqrt{5}R^3)z + (-2 - 7\omega + 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 + 3\sqrt{5})z + (\omega^2 - 1)(3R^2 - 2\sqrt{5}R^2)}{(\omega^2 - 1)(3R^3 - \sqrt{5}R^3)z + (-2 - 7\omega^2 + 3\sqrt{5}\omega^2)}, \quad \text{and} \quad z' = z;$$

$$z' = \frac{(-2\omega - 7 + 3\sqrt{5})z + (\omega - 1)(3R^3 - \sqrt{5}R^3)}{(\omega - 1)(3R^2 - \sqrt{5}R^2)z + (-2 - 7\omega + 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 + 3\sqrt{5})z + (\omega^2 - 1)(3R^3 - \sqrt{5}R^3)}{(\omega^2 - 1)(3R^2 - \sqrt{5}R^2)z + (-2 - 7\omega^2 + 3\sqrt{5}\omega^2)}, \quad \text{and} \quad z' = z;$$

and

$$z' = \frac{(-2\omega - 7 + 3\sqrt{5})z + (\omega - 1)(3R^4 - \sqrt{5}R^4)}{(\omega - 1)(3R - \sqrt{5}R)z + (-2 - 7\omega + 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 + 3\sqrt{5})z + (\omega^2 - 1)(3R^4 - \sqrt{5}R^4)}{(\omega^2 - 1)(3R - \sqrt{5}R)z + (-2 - 7\omega^2 + 3\sqrt{5}\omega^2)},$$

and $z'=z$. The five remaining groups of the third order of the icosahedron are derived from the five sets of opposite triangles which lie on a portion of the surface between those sets which have already been considered.

The first transformation of the third order group corresponding to a counterclockwise rotation of 120° about an axis parallel to the axis of reals is found as follows: the icosahedron is first rotated clockwise about the axis parallel to the axis of imaginaries, sending the pair of opposite triangles to a position where both are parallel to the plane. This is effected by the auxiliary transformation

$$z' = \frac{2z + 3 + \sqrt{5}}{-(3 + \sqrt{5})z + 2} .$$

The solid is then rotated through 120° counterclockwise about an axis perpendicular to the plane. The product transformation of these two rotations is

$$z' = \frac{2\omega z + 3 + \sqrt{5}}{-(3\omega + \sqrt{5}\omega)z + 2} .$$

Finally, the icosahedron is rotated counterclockwise about an axis parallel to the axis of imaginaries by means of the inverse of the third auxiliary transformation,

$$z' = \frac{2z - 3 - \sqrt{5}}{(3 + \sqrt{5})z + 2},$$

making the resulting transformation

$$z' = \frac{(-2\omega z - 7 - 3\sqrt{5})z + (\omega - 1)(3 + \sqrt{5})}{(\omega - 1)(3 + \sqrt{5})z + (-2 - 7\omega - 3\sqrt{5}\omega)}.$$

If the second rotation is through 240°, the result is the second member of this third order group

$$z' = \frac{(-2\omega^2 - 7 - 3\sqrt{5})z + (\omega^2 - 1)(3 + \sqrt{5})}{(\omega^2 - 1)(3 + \sqrt{5})z + (-2 - 7\omega^2 - 3\sqrt{5}\omega^2)}.$$

A complete revolution for the second rotation gives the identity $z' = z$.

These three sets of rotations, when preceded by rotations of each of the fifth roots of unity and succeeded by their respective inverses, will produce all the rotations of corresponding transformations of the remaining third order groups. The third order groups of transformations which correspond to the counterclockwise rotations of each of the remaining sets of opposite faces of the central surface of the icosahedron, as represented in Fig. 1, are then:

$$z' = \frac{(-2\omega - 7 - 3\sqrt{5})z + (\omega - 1)(3R + \sqrt{5}R)}{(\omega - 1)(3R^4 + \sqrt{5}R^4)z + (-2 - 7\omega - 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 - 3\sqrt{5})z + (\omega^2 - 1)(3R + \sqrt{5}R)}{(\omega^2 - 1)(3R^4 + \sqrt{5}R^4)z + (-2 - 7\omega^2 - 3\sqrt{5}\omega^2)},$$

and $z' = z$;

$$z' = \frac{(-2\omega - 7 - 3\sqrt{5})z + (\omega - 1)(3R^2 + \sqrt{5}R^2)}{(\omega - 1)(3R^3 + \sqrt{5}R^3)z + (-2 - 7\omega - 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 - 3\sqrt{5})z + (\omega^2 - 1)(3R^2 + \sqrt{5}R^2)}{(\omega^2 - 1)(3R^3 + \sqrt{5}R^3)z + (-2 - 7\omega^2 - 3\sqrt{5}\omega^2)},$$

and $z' = z$;

$$z' = \frac{(-2\omega - 7 - 3\sqrt{5})z + (\omega - 1)(3R^3 + \sqrt{5}R^3)}{(\omega - 1)(3R^2 + \sqrt{5}R^2)z + (-2 - 7\omega - 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 - 3\sqrt{5})z + (\omega^2 - 1)(3R^3 + \sqrt{5}R^3)}{(\omega^2 - 1)(3R^2 + 5R^2)z + (-2 - 7\omega^2 - 3\sqrt{5}\omega^2)},$$

and $z' = z$;

$$z' = \frac{(-2\omega - 7 - 3\sqrt{5})z + (\omega - 1)(3R^4 + \sqrt{5}R^4)}{(\omega - 1)(3R + \sqrt{5}R)z + (-2 - 7\omega - 3\sqrt{5}\omega)},$$

$$z' = \frac{(-2\omega^2 - 7 - 3\sqrt{5})z + (\omega^2 - 1)(3R^4 + \sqrt{5}R^4)}{(\omega^2 - 1)(3R + \sqrt{5}R)z + (-2 - 7\omega^2 - 3\sqrt{5}\omega^2)},$$

and $z' = z$.

The second and third members of each set can be reproduced from the first member by one and two repetitions, respectively.

Transformations of the second order groups. The first five of the fifteen second order groups to be derived will be those having axes of rotations passing through each pair of midpoints of the upper edges separating the faces of the central surface when the icosahedron is in the position indicated in Fig. 1.

Since the axis parallel to the axis of imaginaries passes through the midpoints of a pair of opposite edges, the transformation, carrying the solid into itself, corresponding to a counterclockwise rotation of 180° is $z' = -1/z$, which is of second order.

Each of the remaining four pairs of opposite edges of the central surface can be brought into the same position as the first pair considered by rotating the solid counterclockwise about an axis perpendicular to the plane through one of the tenth roots of unity. If the solid in each case is then rotated counterclockwise through 180° about the axis parallel to the axis of imaginaries and later rotated about the perpendicular axis through the inverse of the tenth root of unity selected, the remaining second order transformations for the central portion of the icosahedron can be found.

The transformations,

$$z' = Tz \quad \text{or} \quad z' = -R^3 z, \quad z' = -\frac{1}{z}, \quad \text{and} \quad z' = T^9 z \quad \text{or} \quad z' = -R^2 z,$$

in which each of the powers of T represents one of the tenth roots of unity, when performed in the order given, produce $z' = -R/z$, a transformation of order two.

Analogously, the other transformations for rotations about the axes parallel to the plane are

$$z' = -\frac{R^2}{z}, \qquad z' = -\frac{R^3}{z}, \quad \text{and} \quad z' = -\frac{R^4}{z}\cdot$$

It is evident that each of these second order transformations, repeated, will give the identity.

The next five transformations of period two will be those corresponding to the rotations about the axes which pass through each pair of opposite and parallel edges radiating from vertices 1 and 7 of Fig. 1.

In order to find these five transformations, it is first necessary to derive the transformation carrying the icosahedron into itself when the axis of rotation of a pair of opposite edges of the central surface is parallel to the axis of reals.

The general transformation of the sphere when the axis is in this position is

$$z' = \frac{az + b}{bz + a} \cdot$$

So when the vertices are carried into themselves by a counterclockwise rotation of 180°, the transformation is $z' = 1/z$.

To find the second order transformation about an axis passing through the edges determined by the two pairs of vertices (1, 6) and (7, 10), the sphere is first rotated counterclockwise about an axis parallel to the axis of imaginaries by means of the auxiliary transformation,

$$z' = \frac{z - 1}{z + 1} \cdot$$

Next, the sphere is turned about the axis perpendicular to the plane at the origin through the first twentieth root of unity, making the product transformation

$$z' = \frac{Sz - 1}{Sz + 1},$$

where each of the powers of S indicates one of the twentieth roots of unity. The third movement is a counterclockwise rotation of 180° about the axis parallel to the axis of reals, so the single transformation for all the rotations thus far is

$$z' = \frac{\dfrac{S}{z} - 1}{\dfrac{S}{z} + 1} \cdot$$

The last two rotations to be made are the inverses of the first two taken in reverse order, which produce, first

$$z' = \frac{\dfrac{1}{S^{18}z} - 1}{\dfrac{1}{S^{18}z} + 1},$$

and then

$$z' = \frac{-z + 1 - S^{18}z - S^{18}}{-z + 1 + S^{18}z + S^{18}},$$

which, in terms of the fifth roots of unity, is

$$z' = \frac{(-1 + R^2)z + 1 + R^2}{(-1 - R^2)z + 1 - R^2},$$

which brings, with one repetition, the identity, $z' = z$.

The second order transformation corresponding to a counterclockwise rotation of 180° about an axis through opposite edges determined by the two pairs of vertices (1, 5) and (7, 11) can be obtained from the same six transformations used in deriving the second order transformation above, when they are used after and before the transformations,

$$z' = Rz \quad \text{and} \quad z' = R^4 z,$$

respectively. The single second order transformation which corresponds to all six of the transformations may be employed instead. The following transformations, if performed in the order

$$z' = Rz, \qquad z' = \frac{(-1 + R^2)z + 1 + R^2}{(-1 - R^2)z + 1 - R^2}, \quad \text{and} \quad z' = R^4 z,$$

produce the second order transformation

$$z' = \frac{(-1 + R^2)z + R + R^3}{(-R^4 - R)z + 1 - R^2}.$$

Analogously, the remaining three second order transformations which have axes of rotation passing through the opposite edges which radiate from vertices 1 and 7 are

$$z' = \frac{(-1 + R^2)z + R^2 + R^4}{(-R^3 - 1)z + 1 - R^2}, \qquad z' = \frac{(-1 + R^2)z + R^3 + 1}{(-R^2 - R^4)z + 1 - R^2},$$

and

$$z' = \frac{(-1 + R^2)z + R^4 + R}{(-R - R^3)z + 1 - R^2}.$$

The final five second order transformations to be derived are those whose axes of rotation pass through each pair of opposite edges which are parallel to the plane.

The process of finding the second order transformation corresponding to a counterclockwise rotation of 180° about an axis which passes through the opposite pair of edges determined by the pairs of vertices (2, 3) and (8, 9) in Fig. 1 is as follows: the sphere is first rotated counterclockwise about an axis parallel to the axis of imaginaries by means of the transformation

$$z' = \frac{z - 1}{z + 1}.$$

Then the sphere is turned counterclockwise about the perpendicular axis through the first tenth root of unity, making the product transformation for both the rotations thus far

$$z' = \frac{-R^2 z - 1}{-R^3 z + 1},$$

since $-R^3=T$.

If the sphere then is rotated counterclockwise 180° about the axis parallel to the axis of imaginaries, the resulting transformation for all three rotations is

$$z' = \frac{\dfrac{R^3}{z} - 1}{\dfrac{R^3}{z} + 1}.$$

Finally, if the inverse operations of both of the first two transformations are carried out in reverse order, they give

$$z' = \frac{\dfrac{-R}{z} - 1}{\dfrac{-R}{z} + 1}, \quad \text{and then,} \quad z' = \frac{(R-1)z - R - 1}{(R+1)z - R + 1},$$

which gives the identity upon one repetition.

When the axis of rotation passes through opposite edges determined by the pairs of vertices (6, 2) and (8, 10), the corresponding transformation for a counterclockwise rotation of 180° can be found by using the following transformations in the order given:

$$z' = Rz, \quad z' = \frac{z-1}{z+1}, \quad z' = Tz \quad \text{or} \quad z' = -R^3 z, \quad z' = -\frac{1}{z},$$

$$z' = T^9 z \quad \text{or} \quad z' = -R^2 z, \quad z' = \frac{z+1}{-z+1}, \quad \text{and} \quad z' = R^4 z;$$

or, in place of the middle five, their product transformation,

$$z' = \frac{(R-1)z - R - 1}{(R+1)z - R + 1},$$

can be used. The result in either case is

$$z' = \frac{(R-1)z - R^2 - R}{(R^4+1)z - R + 1}.$$

Analogously, then, the remaining second order transformations are:

$$z' = \frac{(R-1)z - R^3 - R^2}{(R^4+R^3)z - R + 1}, \qquad z' = \frac{(R-1)z - R^4 - R^3}{(R^3+R^2)z - R + 1},$$

and

$$z' = \frac{(R-1)z - 1 - R^4}{(R^2+R)z - R + 1}.$$

References

1. Harold Coxeter, The real projective plane, McGraw-Hill, New York, 1949.
2. Karl Doehlemann, Geometrische Transformationen, Leipzig, W. de Gruyter, 1930.
3. Boyd Patterson, Projective geometry, Wiley, New York, 1937.
4. Oswald Veblen and John Young, Projective geometry, Ginn and Co., New York, 1918.

AN ADDITIONAL REMARK CONCERNING THE DEFINITION OF A FIELD

JOSEPH J. MALONE, JR., University of Houston

In this MAGAZINE, A. H. Lightstone [1] has shown that the right-distributive law is not a consequence of the other defining properties of a field. He does this by giving the following example of a nonright-distributive system:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	1
1	0	1.

We shall show that if $(S, +, \cdot)$ is such that
(i) $(S, +)$ is an Abelian group,
(ii) $(S-\{0\}, \cdot)$ is an Abelian group,
(iii) $(S, \cdot)$ is a semigroup,
(iv) multiplication is left-distributive over addition and
(v) the order of S is greater than 2,
then multiplication is right-distributive over addition. Thus Lightstone's example is the only instance of a nonright-distributive "field," i.e., other than in his one example, the right-distributive law is a consequence of the other field postulates.

For $x, y, z \in S$, $(y+z)x$ will equal $yx+zx$ if $y \neq 0$, $z \neq 0$, or $y+z \neq 0$. Postulates (ii) and (iv) yield this result. If $y=0$, $z=0$, or $y+z=0$, then the validity of the right-distributive law depends upon establishing that $0x=0$. Note that the left-distributive law assures us that $x0=0$.

If $x \in S$ and $x \neq 0$, $0x=(x0)x=x(0x)$ and $(0x)x^{-1}=x(0x)x^{-1}$. Thus

$$0e = x(0e). \tag{1}$$

Either $0e=0$ or $0e \neq 0$. If $0e \neq 0$ multiply each side of (1) on the right by the multiplicative inverse of $0e$ and obtain $e=xe$ or $e=x$. This is Lightstone's case in which S contains only 0 and e, but is contrary to our postulate (v). If we then let $0e=0$ and select x so that $x \neq e$, $x \neq 0$, we obtain $0=0e=0(xx^{-1})=(0x)x^{-1}$. Since (ii) assures us that a product is 0 only if one of the factors is 0 and since $x^{-1} \neq 0$, we have $0x=0$. Thus we have demonstrated the desired result and the right-distributive law follows as a consequence of the other field postulates.

Reference

1. A. H. Lightstone, A remark concerning the definition of a field, this MAGAZINE, 37 (1964) 12–13.

References

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A PARTIAL SOLUTION TO A CONJECTURE OF GOLOMB

MARTIN J. COHEN, Beverly Hills, California

The following problem has been proposed by Golomb in [1]: Let M be a m-dimensional $n\times n\times \cdots \times n$ array of nonnegative integers; let $A(a_1, a_2, \cdots, a_m)$ be the entry whose position is $(a_1, a_2, \cdots, a_m)$; and let

$$S(a_1, a_2, \cdots, a_m) = \sum_{k_1=1}^{n} A(k_1, a_2, \cdots, a_m) + \sum_{k_2=1}^{n} A(a_1, k_2, \cdots, a_m) + \cdots + \sum_{k_m=1}^{n} A(a_1, a_2, \cdots, k_m).$$

If $S(a_1, a_2, \cdots, a_m) \geqq n$ whenever $A(a_1, a_2, \cdots, a_m) = 0$, what can we say about the sum of all the entries of M?

Golomb conjectured that this sum is $\geqq n^m/m$ but this has only been proved for $m=2$ (see [1]). In this note it is proved that the sum is $> n^m/(m+1)$.

Let S be the sum of the elements of M and Z the number of zeros in M so that $S+Z \geqq n^m$ and

$$S = \sum_{a_1=1}^{n} \sum_{a_2=1}^{n} \cdots \sum_{a_m=1}^{n} A(a_1, a_2, \cdots, a_m).$$

We see that

$$\sum_{a_1=1}^{n} \sum_{a_2=1}^{n} \cdots \sum_{a_m=1}^{n} S(a_1, a_2, \cdots, a_m) = mnS$$

since, for any i,

$$\sum_{a_1=1}^{n} \sum_{a_2=1}^{n} \cdots \sum_{a_m=1}^{n} \sum_{k_i=1}^{n} A(a_1, \cdots, k_i, \cdots, a_m) = nS.$$

Also, since $S(a_1, \cdots, a_m) \geqq n$ if $A(a_1, \cdots, a_m) = 0$ and $S(a_1, \cdots, a_m) \geqq mA(a_1, \cdots, a_m)$ if $A(a_1, \cdots, a_m) \neq 0$,

$$\sum_{a_1=1}^{n} \cdots \sum_{a_m=1}^{n} S(a_1, \cdots, a_m) \geqq Zn + Sm \geqq S(m-n) + n^{m+1}.$$

Thus $mnS \geqq (m-n)S + n^{m+1}$ or

$$S \geqq \frac{n^{m+1}}{mn+n-m} > \frac{n^m}{m+1}.$$

Note that $S \geqq n^m/m$ if $m \geqq n$.

Reference

1. Solution to problem E 1535, Amer. Math. Monthly, 70 (1963) 1005.

THE DEFINITION OF FUNCTIONS FROM CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS

PAULA WINN CARTER, Southern Methodist University AND
F. MAX STEIN, Colorado State University

1. Introduction. In the study of mathematics various functions are defined or arise in a variety of ways. For example, the reader's initial introduction to the trigonometric functions was probably to consider them as ratios. Other functions may have been introduced as inverses; e.g., the logarithm function may have been defined as the inverse of the exponential function.

On the other hand, the logarithm function may have been defined as an integral, see [2], where

$$\ln x = \int_1^x \frac{dt}{t}, \qquad x > 0,$$

with the property that $\ln(1) = 0$.

Also as one progresses in his study of mathematics he is able to define functions involving parameters that reduce to previously defined functions for certain particular values of the parameters. We cite first as a simple example the function $f(x) = b^{ax}$; the graph of $y = f(x)$ is a straight line for $a = 0$, $b \neq 0$, and is the curve $y = e^x$ for $a = 1$ and $b = e$.

As a second example the gamma function is initially defined in terms of an improper integral,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad x > 0;$$

the definition is then extended to negative nonintegral values of x by the recursion relation $\Gamma(x+1) = x\Gamma(x)$. Furthermore for positive integral values of x, the gamma function takes on the values of the factorial function. Thus the gamma function is often described as the generalized factorial function.

In this paper we propose first to present some functions that have been defined as solutions of certain nonlinear differential equations. Then we shall show how, for certain values of the parameter involved, these new functions reduce to old friends—some functions that have been defined by other means.

2. The Jacobian elliptic functions. The first Jacobian elliptic function $\mathrm{sn}(k, t)$ has been defined (see [1]) as the inverse of the elliptic function,

$$(1) \quad t = \int_0^y [(1 - x^2)(1 - k^2x^2)]^{-1/2}dx; \quad \text{i.e.,} \quad t = \mathrm{sn}^{-1}\, y \quad \text{and} \quad y = \mathrm{sn}(k, t),$$

where $\mathrm{sn}(k, t)$ is a function of the variable t and the parameter k, called the modulus, and $0 < k < 1$.

That is, $y = \mathrm{sn}(k, t)$ is required to satisfy the differential equation

$$(2) \qquad (y')^2 = (1 - y^2)(1 - k^2y^2),$$

with the conditions that $y(0) = 0$ and $y'(0) > 0$.

If in (2) we make the substitution $y^2=1-x^2$, we get the equation

$$(y')^2 = (1-y^2)[1-k^2(1-y^2)], \qquad y(0) = 1, \tag{3}$$

after changing notation from x to y. We now define the function $y=\text{cn}(k, t)$ to be the solution of (3) and note that $\text{sn}(k, t)$ and $\text{cn}(k, t)$ satisfy the identity

$$\text{sn}^2(k, t) + \text{cn}^2(k, t) = 1.$$

If in (2) we make the substitution $y^2=(1-x^2)/k^2$, we obtain the equation

$$(y')^2 = (1-y^2)(y^2+k^2-1), \qquad y(0) = 1, \tag{4}$$

again after changing notation from x to y. The function $y=\text{dn}(k, t)$ is defined to be the solution of (4); we see from the substitution made that $\text{sn}(k, t)$ and $\text{dn}(k, t)$ satisfy the identity

$$\text{dn}^2(k, t) = 1 - k^2 \,\text{sn}^2(k, t).$$

Having defined the three main Jacobian elliptic functions $\text{sn}(k, t)$, $\text{cn}(k, t)$, and $\text{dn}(k, t)$, we proceed to define the corresponding reciprocal and quotient functions, using Glaisher's notation (see [**4**], p. 494):

$$\text{ns}(k, t) = [\text{sn}(k, t)]^{-1}, \qquad \text{nc}(k, t) = [\text{cn}(k, t)]^{-1},$$

$$\text{sc}(k, t) = \frac{\text{sn}(k, t)}{\text{cn}(k, t)}, \qquad \text{cs}(k, t) = \frac{\text{cn}(k, t)}{\text{sn}(k, t)}.$$

These functions satisfy the following differential equations:

(5a) $y = \text{ns}(k, t)$: $(y')^2 = (y^2-1)(y^2-k^2)$, $y(0) = \infty$,

(5b) $y = \text{nc}(k, t)$: $(y')^2 = (y^2-1)[(1-k^2)y^2+k^2]$, $y(0) = 1$,

(5c) $y = \text{sc}(k, t)$: $(y')^2 = (1+y^2)[1+(1-k^2)y^2]$, $y(0) = 0$,

(5d) $y = \text{cs}(k, t)$: $(y')^2 = (1+y^2)(1+y^2-k^2)$, $y(0) = \infty$.

The Jacobian elliptic functions are defined for $0<k<1$. When $k=0$, however, it is known that (see Bowman, [**1**], p. 10–11) the elliptic functions reduce to the circular trigonometric functions; i.e.,

$$\text{sn}(0, t) = \sin t, \qquad \text{cn}(0, t) = \cos t, \qquad \text{dn}(0, t) = 1,$$

$$\text{sc}(0, t) = \tan t, \qquad \text{cs}(0, t) = \cot t,$$

$$\text{ns}(0, t) = \csc t, \qquad \text{nc}(0, t) = \sec t,$$

which may be shown by letting $k=0$ in the corresponding differential equations.

When $k=1$, it is known that (see Bowman, [**1**], p. 10–11) the elliptic functions reduce to the hyperbolic functions; i.e.,

$$\text{sn}(1, t) = \tanh t, \qquad \text{cn}(1, t) = \text{dn}(1, t) = \text{sech}\, t,$$

$$\text{ns}(1, t) = \coth t, \qquad \text{nc}(1, t) = \cosh t,$$

$$\text{sc}(1, t) = \sinh t, \qquad \text{cs}(1, t) = \text{csch}\, t,$$

which may be shown by letting $k=1$ in the corresponding differential equations.

When k is allowed to become imaginary, we define an extension of these functions. Letting $k=i\mu$, $k^2=-\mu^2$, we see that the differential equation (2) becomes

$$(y')^2 = (1-y^2)(1+\mu^2y^2), \qquad y(0) = 0. \tag{6}$$

We then define the function $y=\text{hsn}(\mu, t)$ as the solution of (6). (The notation $\text{hsn}(\mu, t)$ is chosen in anticipation of the analogy to a corresponding function to be defined later.)

Upon making the substitution $y^2=1-x^2$ in (6), we obtain the differential equation

$$(y')^2 = (1-y^2)[1+\mu^2(1-y^2)], \qquad y(0) = 1, \tag{7}$$

after changing notation from x to y. The function $y=\text{hcn}(\mu, t)$ is defined to be the solution of (7), and we see further that $\text{hsn}(\mu, t)$ and $\text{hcn}(\mu, t)$ satisfy the identity $\text{hsn}^2(\mu, t)+\text{hcn}^2(\mu, t)=1$.

If in (6) we make the substitution $y^2=(x^2-1)/\mu^2$, we obtain the differential equation

$$(y')^2 = (1-y^2)(y^2-\mu^2-1), \qquad y(0) = 1, \tag{8}$$

after changing notation from x to y. The function $y=\text{hdn}(\mu, t)$ is defined to be the solution of (8), and we see from the substitution made that $\text{hdn}(\mu, t)$ and $\text{hsn}(\mu, t)$ satisfy the identity

$$\text{hdn}^2(\mu, t) = 1 + \mu^2\,\text{hsn}^2(\mu, t).$$

Having defined the three functions $\text{hsn}(\mu, t)$, $\text{hcn}(\mu, t)$, and $\text{hdn}(\mu, t)$, we proceed as before to define the reciprocal and quotient functions:

$$\text{hns}(\mu, t) = [\text{hsn}(\mu, t)]^{-1}, \qquad \text{hnc}(\mu, t) = [\text{hcn}(\mu, t)]^{-1},$$

$$\text{hsc}(\mu, t) = \frac{\text{hsn}(\mu, t)}{\text{hcn}(\mu, t)}, \qquad \text{hcs}(\mu, t) = \frac{\text{hcn}(\mu, t)}{\text{hsn}(\mu, t)}.$$

These functions satisfy the following differential equations:

$$y = \text{hns}(\mu, t):\ (y')^2 = (y^2-1)(y^2+\mu^2), \qquad y(0) = \infty, \tag{9a}$$

$$y = \text{hnc}(\mu, t):\ (y')^2 = (y^2-1)[(1+\mu^2)y^2-\mu^2], \qquad y(0) = 1, \tag{9b}$$

$$y = \text{hsc}(\mu, t):\ (y')^2 = (y^2+1)[(1+\mu^2)y^2+1], \qquad y(0) = 0, \tag{9c}$$

$$y = \text{hcs}(\mu, t):\ (y')^2 = (1+y^2)(1+y^2+\mu^2), \qquad y(0) = \infty. \tag{9d}$$

As is to be expected, when $\mu=0$ these functions reduce to the trigonometric functions, $\text{hsn}(0, t)=\sin t$, $\text{hcn}(0, t)=\cos t$, and so on, as in the case for $k=0$. This may be shown by letting $\mu=0$ in the differential equations (6), (7), and (9a, b, c, d).

Thus the Jacobian elliptic functions and the corresponding elliptic functions for the imaginary modulus k are examples of functions which may be defined as the solutions of certain nonlinear differential equations, which reduce to the circular trigonometric functions and the hyperbolic functions for certain values of the parameter k.

3. The quasi-trigonometric functions. We now consider the differential equation

$$(y')^2 = m^{-2}(1 - m^2y^2), \qquad y(0) = 0, \qquad y'(0) > 0. \tag{10}$$

We define the function $y = \text{nis}\ \theta$ to be the solution of (10), where m is a parameter such that $m = \sin\lambda$, $0 < \lambda \leqq \pi/2$. Since $y'(0) > 0$ and $|y| \leqq 1/m$, (10) reduces to

$$y' = (m^{-2} - y^2)^{1/2}, \quad \text{or} \quad \int_0^y (m^{-2} - t^2)^{-1/2} dt = \theta.$$

Thus $\theta = \sin^{-1}(my)$ and

$$y = \frac{\sin\theta}{\sin\lambda} = \text{nis}\ \theta. \tag{11}$$

We then consider the differential equation (10) with different boundary conditions,

$$(y')^2 = m^{-2}(1 - m^2y^2), \qquad y(0) = 1, \qquad |y| \leqq 1/m. \tag{12}$$

We define the function $y = \text{conis}\ \theta$ to be the solution of (12). Then $y' = m^{-1}(1 - m^2y^2)$

$$\theta = \int_y^{\prime} (m^{-2} - t^2)^{-1/2} dt = \cos^{-1} my - \cos^{-1} m,$$

or $y = \cos\theta - \cos\lambda\ \text{nis}\ \theta$; i.e.,

$$\text{conis}\ \theta = \cos\theta - \cos\lambda\ \text{nis}\ \theta. \tag{13}$$

From the functions nis θ and conis θ the following reciprocal and quotient functions may be defined:

$$\begin{aligned} \text{coces}\ \theta &= (\text{nis}\ \theta)^{-1}, & \text{ces}\ \theta &= (\text{conis}\ \theta)^{-1}, \\ \text{nat}\ \theta &= \frac{\text{nis}\ \theta}{\text{conis}\ \theta}, & \text{conat}\ \theta &= \frac{\text{conis}\ \theta}{\text{nis}\ \theta}. \end{aligned} \tag{14}$$

These functions satisfy the following differential equations:

(15a) $y = \text{coces}\ \theta$: $(y')^2 = m^{-2}y^2(y^2 - m^2)$, $y(0) = \infty$,

(15b) $y = \text{ces}\ \theta$: $(y')^2 = m^{-2}y^2(y^2 - m^2)$, $y(0) = 1$,

(15c) $y = \text{nat}\ \theta$: $(y')^2 = m^{-2}[y^2 + 2y(1 - m^2)^{1/2} + 1]^2$, $y(0) = 0$,

(15d) $y = \text{conat}\ \theta$: $(y')^2 = m^{-2}[y^2 + 2y(1 - m^2)^{1/2} + 1]^2$, $y(0) = \infty$.

When $\lambda = \pi/2$, $m = 1$ so that these functions reduce to the conventional trigonometric functions. This may be shown by letting $\lambda = \pi/2$ in equations (11) and (13), so that nis $\theta = \sin\theta$ and conis $\theta = \cos\theta$; the reciprocal and quotient functions follow so that coces $\theta = \csc\theta$, ces $\theta = \sec\theta$, nat $\theta = \tan\theta$, and conat $\theta = \cot\theta$.

That these functions reduce to the trigonometric functions may also be shown by letting $m=1$ in the differential equations (10), (12), and (15a, b, c, d).

If λ is allowed to assume imaginary values, we further extend our class of functions. Letting $\lambda = i\phi$, where ϕ is real and $m=\sinh\phi$, we obtain from the differential equation (10)

$$(y')^2 = m^{-2}(1+m^2y^2), \qquad y(0) = 0. \tag{16}$$

The function $y=\text{hnis}\,\theta$ is defined to be the solution of (16). In a manner analogous to that in which the function nis θ was shown to equal $\csc\lambda\sin\theta$, it may be shown that

$$\text{hnis}\,\theta = \frac{\sinh\theta}{\sinh\phi}\cdot \tag{17}$$

The function $y=\text{hconis}\,\theta$ is defined to be the solution of the differential equation

$$(y')^2 = m^{-2}(1+m^2y^2), \qquad y(0) = 1. \tag{18}$$

In a manner analogous to that in which the function conis θ was shown to equal $(\cos\theta - \cos\lambda\ \text{nis}\,\theta)$, it may be shown that

$$\text{hconis}\,\theta = \cosh\theta - \coth\phi\sinh\theta. \tag{19}$$

Again in like manner the reciprocal and quotient functions may be defined:

$$\text{hcoces}\,\theta = (\text{hnis}\,\theta)^{-1}, \qquad \text{hces}\,\theta = (\text{hconis}\,\theta)^{-1},$$

$$\text{hnat}\,\theta = \frac{\text{hnis}\,\theta}{\text{hconis}\,\theta}, \qquad \text{hconat}\,\theta = \frac{\text{hconis}\,\theta}{\text{hnis}\,\theta}\cdot$$

These functions satisfy the following differential equations:

$$\text{(20a)}\quad y = \text{hcoces}\,\theta: \quad (y')^2 = m^{-2}y^2(y^2+m^2), \qquad y(0) = \infty,$$

$$\text{(20b)}\quad y = \text{hces}\,\theta: \quad (y')^2 = m^{-2}y^2(y^2+m^2), \qquad y(0) = 1,$$

$$\text{(20c)}\quad y = \text{hnat}\,\theta: \quad (y')^2 = m^{-2}[y^2+2y(1+m^2)^{1/2}+1]^2, \qquad y(0) = 0,$$

$$\text{(20d)}\quad y = \text{hconat}\,\theta: \quad (y')^2 = m^{-2}[y^2+2y(1+m^2)+1]^2, \qquad y(0) = \infty.$$

Thus we have another example of a class of functions which may be defined from certain nonlinear differential equations, and which reduce to the conventional trigonometric functions for certain values of the parameter.

4. Discussion of the quasi-trigonometric functions. It should be observed that the class of functions defined in section three above is the same as that defined as quasi-trigonometric functions by Strand and Stein in [3]. There the quasi-trigonometric functions are defined in a manner analogous to that of conventional trigonometry using an oblique coordinate system, Fig. 1, where $0 \leqq \lambda \leqq 2\pi$, x is the abscissa, y the ordinate, and n the distance from the origin to the point $P(x, y)$. The present definitions (11), (13), and (14) are there derived from geometric ratios; certain basic identities involving the quasi-functions and the trigonometric functions are also derived, all of which reduce to conventional trigonometric functions when $\lambda=\pi/2$.

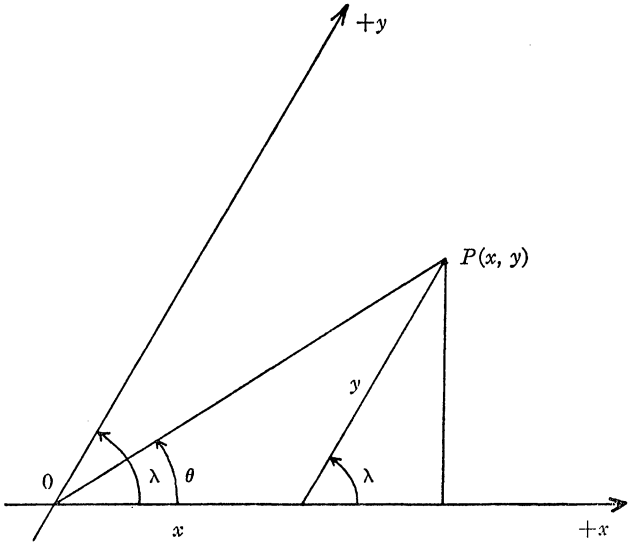

FIG. 1

When $0<\lambda\leqq\pi/2$, i.e., $0<m\leqq 1$, the quasi-trigonometric functions are said to be quasi-elliptic in nature, [**3**], a situation analogous to the Jacobian elliptic functions for $0<k<1$. When the parameter k is allowed to assume certain values, the Jacobian elliptic functions reduce to the hyperbolic functions; in quasi-trigonometry, functions referred to as quasi-hyperbolic [**3**] result when the parameter is allowed to become imaginary.

5. Conclusion. Here we have shown how Jacobian elliptic functions and quasi-trigonometric functions may be defined as solutions of certain nonlinear differential equations. Also shown is how these functions reduce to the well-known trigonometric or hyperbolic functions for particular values of the parameters involved. Thus the functions here defined may be considered as generalizations of the trigonometric and hyperbolic functions.

As a side result we observe the tenuous connection between all of the functions involved, a partial answer to the question often asked by students when they encounter the hyperbolic functions for the first time.

Prepared in a National Science Foundation Undergraduate Science Education Program in Mathematics at Colorado State University by Miss Carter under the direction of Professor Stein.

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WAITING FOR A BUS

ALAN SUTCLIFFE, Cheshire, England

Suppose two independent bus services operate hourly over a route, how long can an intending passenger, who is ignorant of the timings, expect to wait?

Taking the time of one of the buses as 0, 1, 2, $\cdots$ hours, and the other as $x, x+1, x+2, \cdots$ hours, we see that the passenger has a chance of x of arriving between 0 and x, when he will have an average waiting time of $\frac{1}{2}x$, and a chance of $1-x$ of arriving between x and 1, with an average waiting time of $\frac{1}{2}(1-x)$. This is shown diagrammatically in Figure 1. Thus his combined waiting time will be $\frac{1}{2}x^2+\frac{1}{2}(1-x)^2=x^2-x+\frac{1}{2}$, in hours. Now x may have any value from 0 to 1, so that the average waiting time over all these values is

$$\int_0^1 (x^2 - x + \tfrac{1}{2})dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x\right]_0^1 = \tfrac{1}{3}, \text{ in hours.}$$

These 20 minutes of waiting time compare with 15 minutes in the case of a coordinated service with the optimum half hour interval.

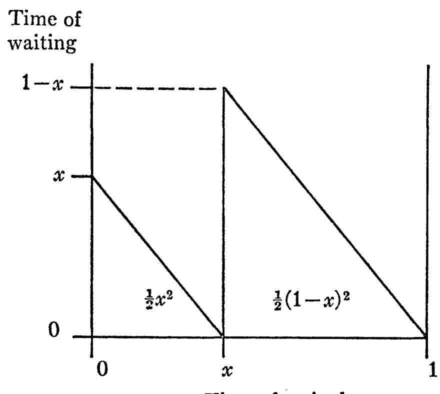

FIG. 1.

If N independent hourly services operate, having times $x_0=0\leqq x_1\leqq x_2 \cdots \leqq x_{N-1}\leqq 1$, then the situation is as shown in Figure 2. For particular values of the x_r, the average waiting time will be

$$\sum_{r=0}^{N-1} \tfrac{1}{2}(x_{r+1} - x_r)^2, \text{ taking } x_N = 1.$$

Hence, as the x_r can vary, it is necessary to integrate for each one, and the average waiting time of x_N hours is given by

$$X_N = (N-1)!\int_0^1\int_0^{x_{N-1}} \cdots \int_0^{x_2} \sum_{r=0}^{N-1} \frac{1}{2}(x_{r+1} - x_r)^2 dx_1 \cdots dx_{N-2}dx_{N-1}.$$

Then, writing

$$S_N\{z\} \quad \text{for} \quad (N-1)!\int_0^1\int_0^{x_{N-1}}\cdots\int_0^{x_2}\{z\}\,dx_1\cdots dx_{N-2}dx_{N-1},$$

$$X_N = S_N\left\{\sum_{n=1}^{N-1} x_n^2 - \sum_{n=1}^{N-2} x_n x_{n+1} - x_{N-1} + \frac{1}{2}\right\}.$$

Applying the repeated integrations to the terms separately gives

$$S_N\left\{\sum_{n=1}^{N-1} x_n^2\right\} = \frac{N-1}{3}, \quad S_N\left\{\sum_{n=1}^{n-2} x_n x_{n+1}\right\} = \frac{(N-2)(N-1)(2N+3)}{6N(N+1)}, \quad \text{and}$$

$$S_N\left\{-x_{N-1} + \frac{1}{2}\right\} = -\frac{N-1}{N} + \frac{1}{2}.$$

Hence $X_N = 1/(N+1)$. This compares with an expected waiting time of $1/(2N)$ hours if the buses were equally spaced throughout each hour.

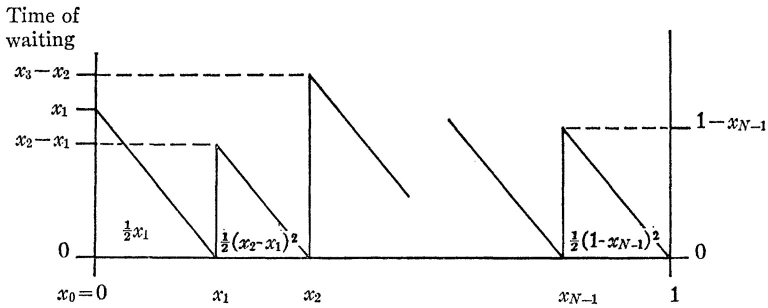

FIG. 2.

In the case of N independently operated buses, each with an N-hourly service, the expected waiting time is $N/(N+1)$, compared with $\frac{1}{2}$ hour for the corresponding regular service. Hence, in the limit $N\to\infty$, a completely random service leaves passengers waiting just twice as long as a regular service using the same number of buses.

To end with a more cheerful thought: provided only that an intending passenger is on a bus route, the time he will most probably have to wait for a bus is zero, or very nearly, since on some routes, at some times, there is a continuous procession of buses and virtually no waiting during these times.

A REAL-LIFE APPLICATION OF MATHEMATICAL SYMBOLISM

R. M. REDHEFFER, University of California at Los Angeles

Much of the power of any mathematical notation is due to its *terseness*, a quality that is well known to mathematicians and is constantly used in their researches. Not so well known, perhaps, is the ready applicability of this same

$$S_N\{z\} \quad \text{for} \quad (N-1)!\int_0^1\int_0^{x_{N-1}}\cdots\int_0^{x_2}\{z\}dx_1\cdots dx_{N-2}dx_{N-1},$$

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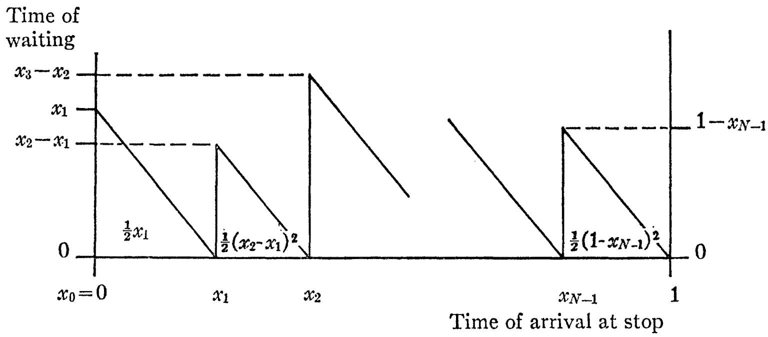

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Much of the power of any mathematical notation is due to its *terseness*, a quality that is well known to mathematicians and is constantly used in their researches. Not so well known, perhaps, is the ready applicability of this same quality to the communication problems of everyday life. In illustration of this remark, we present the following fundamental

THEOREM. *The expression*

$$0.5\ rd/\tilde{B}$$

describes what happens when the highway is covered with certain members of the order Coleoptera.

For proof, write the expression in the form

$$0.5r\,\frac{1}{a\tilde{B}/(da)}, \tag{1}$$

and consider the fraction forming the denominator. Now, $a\tilde{B}$ is pronounced "a B tildah" and hence, in the formula $a\tilde{B}/(da)$ the last syllable of the "tilde" cancels with the "*da*" of the denominator. That is,

$$\frac{a\tilde{B}}{da} = \text{a beetle}. \tag{2}$$

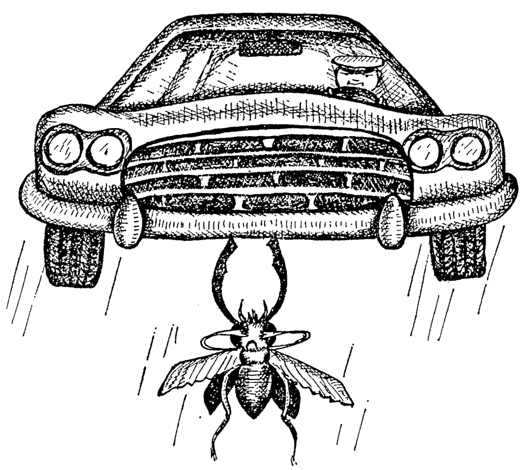
Substituting (2) in (1) yields the first member in the following chain of equalities; the others should cause no difficulty:

$$\begin{aligned} 0.5r\,\frac{1}{a\tilde{B}/(da)} &= 0.5r \text{ one over a beetle} \\ &= 0.5r \text{ wun over a beetle} \\ &= 0.5w \text{ run over a beetle} \\ &= 0.5 \text{ double (you run over a beetle).} \end{aligned}$$

An obvious simplification gives the final answer,

You run over a beetle!

when we set $y = Y$ and take the factorial.



AN ELEMENTARY PROOF OF A THEOREM OF HERSTEIN

JIANG LUH, Indiana State College

Let R be a ring and let Z be its center, i.e., the set of all elements of R which commute with every element of R. It is well known that Z is a subring of R.

In 1951, Herstein (e.g., [2, 3]) proved that if, for each $x\in R$, there exists an integer $n(x)>1$ such that $x^{n(x)}+x\in Z$, then R is commutative. This result was established by essential use of Zorn's lemma. The author was faced with the problem of presenting this celebrated result to undergraduate students in their first course in abstract algebra. The following approach was used: First a reference was made to a recent paper in which Raymond and Christine Ayoub [1] have supplied a simple proof of the following

LEMMA 1. *If, for every $x\in R$, $x^2+x\in Z$, then R is commutative.*

For easy reference, we exhibit their proof here.

Proof of Lemma 1. For any $x, y\in R$, $(x+y)^2+(x+y)\in R$. By expanding and noting that $x^2+x\in R$, $y^2+y\in R$, we have $xy+yx\in Z$. Hence, $x(xy+yx) =(xy+yx)x$, or $x^2y=yx^2$. Thus, $x^2\in Z$ and therefore $x\in Z$.

An elementary proof of the Herstein theorem when $n(x)=3$ for every $x\in R$ was then presented and is exhibited here.

THEOREM. *If $x^3+x\in Z$ for every $x\in R$, then R is commutative.*

We begin with

LEMMA 2. *If $x^3+x\in Z$ for every $x\in R$, then $2x\in Z$ for every $x\in R$.*

Proof. Let x and y be arbitrary elements of the ring R. Then $(x+y)^3+(x+y) \in Z$, and $(x-y)^3+(x-y)\in Z$. Expanding and noting that both x^3+x and y^3+y lie in Z, we have $x^2y+xyx+yx^2+y^2x+yxy+xy^2\in Z$, and $x^2y+xyx+yx^2-y^2x -yxy-xy^2\in Z$. Thus, their sum, $2(x^2y+xyx+yx^2)$, lies in Z. Hence $x[2(x^2y +xyx+yx^2)]=[2(x^2y+xyx+yx^2)]x$, or $2x^3y=2yx^3$. This implies that $2x^3\in Z$, so that $2x=2(x^3+x)-2x^3\in Z$.

LEMMA 3. *If $x^3+x\in Z$ for every $x\in R$, then $x^5+x^4\in Z$ for every $x\in R$.*

Proof. Let x be an arbitrary element of R. Then $(x^2+x)^3+(x^2+x)\in Z$, or $x^6+3x^5+3x^4+x^3+x^2+x\in Z$. Using the fact that $x^6+x^2\in Z$ and that $x^3+x\in Z$, we have $3(x^5+x^4)\in Z$. But, by Lemma 2, $2(x^5+x^4)\in Z$. Hence, $x^5+x^4\in Z$.

LEMMA 4. *If $x^3+x\in Z$ for every $x\in R$, then $x^9+x^6\in Z$ for every $x\in R$.*

Proof. Let x be an arbitrary element of R. By Lemma 3, $(x^2+x)^5+(x^2+x)^4 \in Z$, or $x^{10}+5x^9+10x^8+10x^7+5x^6+x^5+x^8+4x^7+6x^6+4x^5+x^4\in Z$. Since, by Lemma 3, $x^{10}+x^8\in Z$, $x^5+x^4\in Z$, and by Lemma 2, $10(x^8+x^7)+4(x^7+x^5)+6x^6 \in Z$, we obtain that $5(x^9+x^6)\in Z$. Using Lemma 2 again, we have $x^9+x^6\in Z$.

Proof of the theorem. For arbitrary $x\in R$, $x^2+x=(x^6+x^2)+(x^9+x^3)-(x^9+x^6) -(x^3+x)+2x$. According to Lemma 2 and Lemma 4, we see that $x^2+x\in Z$. Hence, by Lemma 1, R is commutative.

Remark. It would be interesting to prove Herstein theorem using elementary methods for other values of n. However, some new technique seems to be needed to take care of all cases.

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COMMENT ON "NOTE ON CONSECUTIVE INTEGERS WHOSE SUM OF SQUARES IS A PERFECT SQUARE"

J. A. H. HUNTER, Toronto, Ontario

Stanton Philipp's most interesting note (this MAGAZINE, 37 (1964) 218–220) was unfortunately marred by a numerical mistake in the treatment of $n=457$. Having derived $y^2-457u^2=-19\cdot 916$, he concludes:

From the simple continued fraction for $\sqrt{457}$, *we can calculate that* $583531623^2-457\cdot 27296458^2=-19$; *by direct trial, we find that* $5644^2-457\cdot 264^2=916$. *It follows that* (15) *has the solution*

$$y = 5644\cdot 583531623 + 457\cdot 264\cdot 27296458$$

$$u = 264\cdot 583531623 + 5644\cdot 27296458.$$

In fact, $2822^2-457\cdot 132^2=916$. Furthermore, since 916 contains 2^2, one would solve $Y^2-457W^2=-19\cdot 229$. Since $1411^2-457\cdot 66^2=229$, the obviously more *manageable* solution of this will be:

$$Y = 1411\cdot 583531623 - 457\cdot 66\cdot 27296458 = 47353857$$

$$W = -66\cdot 583531623 + 1411\cdot 27296458 = 2215120.$$

Also, that reference to "direct trial" seems open to mild criticism. We have $Y_1^2-457W_1^2=229$, so $Y_1^2\equiv 229\equiv 1600 \pmod{457}$, hence $Y_1=457k\pm 40$, which gives $W_1^2=457k^2\pm 80k+3$. Immediately, with $k=3$ we have the acceptable minimal solution $W_1=66$. This seems preferable to tedious "direct trial."

A NEW TWIST TO AN OLD PROBLEM

DONALD V. WEYERS, Bowling Green State University

In the usual proof of the irrationality of $\sqrt{2}$ by contradiction it is assumed that $\sqrt{2}=a/b$ where a and b are relatively prime integers with $b\neq 0$. From $2b^2=a^2$ we get that 2 is a factor of a^2, etc. It may also be observed, however, that $b\mid a\cdot a$. Now apply the well-known theorem that if $(a, c)=1$ and $c\mid ad$, then $c\mid d$ to conclude that $b\mid a$. Hence, since $(a, b)=1$ it follows that $b=1$ and $a^2=2$ Since $1^2=1$ and $a^2>2$ for $a>1$, we have a contradiction.

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A NOTE ON DINI'S THEOREM

SHAW MONG, National Taiwan University, China

A general form of Dini's theorem states that: If a monotone (uniformly) sequence of continuous real-valued functions $\{f_i\}$ is defined on a compact space X and converges pointwise to a continuous real-valued function f, then it converges uniformly to f.

We shall give a proof of this theorem under a somewhat weaker monotony condition on $\{f_i\}$.

THEOREM (Dini's theorem). *If X is a compact space, and if $\{f_i\}$ is a sequence of continuous real-valued functions defined on X, which converges pointwise to a continuous real-valued function f, and if for each point of a dense subset of X, $\{f_i(a)\}$ $i=1, 2, \cdots$ is a monotone sequence, then $\{f_i\}$ converges uniformly to f.*

The following lemma will enable us to prove this theorem:

LEMMA. *If X is a topological space, and $\{f_i\}$ is a sequence of continuous real-valued functions defined on X, which has the following property on a dense subset A of X: for each $a \in A$, either all $\{f_i(a)\}$ $i=1, 2, \cdots$, are nonpositive or all are nonnegative, then $\{f_i\}$ also has that property on X.*

Proof. Let $P = \{p \in X | f_i(p) \geqq 0, i = 1, 2, \cdots \}$, $N = \{q \in X | f_i(q) \leqq 0, i=1, 2, \cdots \}$. Then $A \subset P \cup N$. The lemma is proved if we can show that for each $x \in X \Rightarrow x \in P \cup N$. This is indeed true, for otherwise we obtain a contradiction: If $x \notin P \cup N$, then there is a pair of positive integers j, k, such that $f_j(x) > 0$ and $f_k(x) < 0$. Since f_j and f_k are continuous, there exists, for any integer n, a neighborhood $U_n(x)$, such that $|f_j(x) - f_j(x')| < 1/n$, $|f_k(x) - f_k(x')| < 1/n$ whenever $x' \in U_n(x)$. By the denseness of A, there always exists an $a_n \in A$, such that $a_n \in U_n(x)$. For $n=1, 2, 3, \cdots$ we obtain a sequence $\{a_n\} \in A$ such that

$$|f_i(x) - f_j(a_n)| < \frac{1}{n}, \quad |f_k(x) - f_k(a_n)| < \frac{1}{n}, \quad \text{for all } n = 1, 2, \cdots.$$

Since $\{a_n\} = (\{a_n\} \cap P) \cup (\{a_n\} \cap N)$, either $\{a_n\} \cap P$ or $\{a_n\} \cap N$ is a subsequence of $\{a_n\}$. Let $\{a_n\} \cap P = \{b_1, b_2, \cdots \}$ (or $\{a_n\} \cap N = \{b_1, b_2, \cdots \}$). Then $f_j(x)$, $f_k(x)$ are the limits of the nonnegative (resp. nonpositive) sequences $\{f_j(b_1), f_j(b_2), \cdots, \}$, $\{f_k(b_1), f_k(b_2), \cdots \}$ respectively, and therefore $f_j(x) \geqq 0$, $f_k(x) \geqq 0$ (or $f_j \leqq 0$, $f_n \leqq 0$, respectively). This contradicts the assumption that $f_j(x) > 0$, and $f_k(x) < 0$.

Proof of theorem. Let $g_i = f_i - f_{i+1}$; then g_i is also a continuous real-valued function on X. Moreover, the sequence $\{g_i\}$ satisfies the condition of the preceding lemma. Hence, for each x of X, either all $\{g_i(x)\}$ are nonnegative or all are nonpositive, i.e., $\{f_i(x)\}$ is a monotone sequence for each x of X. If $\{f_i(x)\}$ is an increasing sequence (or decreasing respectively), then $f(x) \geqq f_i(x)$ for all $i=1, 2, \cdots$ (or $f(x) \leqq f_i(x)$ respectively). In both cases, it is true that $|f(x) - f_i(x)| \leqq |f(x) - f_j(x)|$ for all $i \geqq j$.

The following part can be found in any textbooks dealing with the classical form of this theorem: For given $\epsilon > 0$ and each $x \in X$, there is a positive integer n_x such that as $n \geqq n_x$, $|f(x) - f_n(x)| < \epsilon/3$. Since f and f_{n_x} are continuous, there is a neighborhood $U(x)$ of x such that

$$|f(x) - f(x')| < \frac{\epsilon}{3}, \qquad |f_{n_x}(x) - f_{n_x}(x')| < \frac{\epsilon}{3}, \quad \text{for} \quad x' \in U(x).$$

By assumption, X is compact. Hence, there are a finite number of points x_i in X such that the $U(x_i)$ cover X. Now let n_0 be the maximum of $\{n_{x_i}\}$. Then for any x in X, x belongs to some $U(x_i)$; as $n \geqq n_0$, we obtain

$$|f(x) - f_n(x)| \leqq |f(x) - f_{n_{x_i}}(x)| \leqq |f(x) - f(x_i)| + |f(x_i) - f_{n_{x_i}}(x_i)|$$
$$+ |f_{n_{x_i}}(x_i) - f_{n_{x_i}}(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

i.e., $\{f_i\}$ converges uniformly to f.

A FALLACY IN DIFFERENTIABILITY

ALBERT WILANSKY, Lehigh University

The following argument, which at first glance seems to prove that every differentiable function has a continuous derivative, is offered as a challenge for undergraduates.

Let f be differentiable. Fix x. For $k \neq 0$, let

$$\epsilon = \frac{f(x+k) - f(x)}{k} - f'(x).$$

Then

(1) $$f(x+k) = f(x) + kf'(x) + k\epsilon, \qquad \epsilon \to 0 \text{ as } k \to 0.$$

In (1) take successively $x=0$, $k=2h$; $x=0$, $k=h$; $x=k=h$. This yields

(2) $$f(2h) = f(0) + 2hf'(0) + 2h\epsilon_1$$

(3) $$f(h) = f(0) + hf'(0) + h\epsilon_2$$

(4) $$f(2h) = f(h) + hf'(h) + h\epsilon_3.$$

Solving (4) yields

$$f'(h) = \frac{f(2h) - f(h)}{h} - \epsilon_3.$$

Substitution from (2), (3) yields $f'(h) = f'(0) + 2\epsilon_1 - \epsilon_2 - \epsilon_3$. Hence $f'(h) \to f'(0)$ as $h \to 0$.

form of this theorem: For given $\epsilon > 0$ and each $x \in X$, there is a positive integer n_x such that as $n \geqq n_x$, $|f(x) - f_n(x)| < \epsilon/3$. Since f and f_{n_x} are continuous, there is a neighborhood $U(x)$ of x such that

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TRIPLE PRODUCT TABLES

R. A. JACOBSON, South Dakota State University

Many times an abstract binary operation in a set S is defined by means of a product table. Most properties of the system, with the exception of associativity, are clearly displayed in such a table. In this note, we wish to present a quick and simple method for constructing a triple product table so that the associative properties of the system are also readily visualized.

	a	b	c
a	a	c	b
b	c	b	a
c	b	b	c

TABLE 1

Since the method is easily generalized, we shall restrict our discussion to the set $\{a, b, c\}$ with products as given in Table 1. The Triple Product Table is constructed as follows: Place each element of the set in row one. Directly below each element, y, in the top row we enter both the row and column associated with that y as found in Table 1. These are in reality the products yz and xy, respectively. Columns (rows) from Table 1 are then inserted in the bottom row of boxes, the columns (rows) being the ones associated with the entries in the single row (column) immediately above. For example, the first square in the final array contains the columns found below the elements a, c, b, respectively, in Table 1. Similarly, the last square of the array contains the rows to the right of elements b, a, c, respectively, in Table 1. The bottom array thus displays all triple products $x(yz)$ and $(xy)z$ in adjacent boxes so that associativity can be readily checked. For example, the triple products $c(ab)$ and $(ca)b$ are found in the third row and second column of the two left most boxes in the final array. It is evident that in our system only the triples axa, xxx, ccx, cba, and cbb are associative.

a						b						c					
a	c	b		a						c						b	
				c		c	b	a		b						a	
				b						b		b	b	c		c	
	(az)			(xa)			(bz)			(xb)			(cz)			(xc)	
a	b	c	a	c	b	b	c	a	b	b	c	c	c	b	c	b	a
c	a	b	b	b	c	a	b	c	c	b	a	b	b	a	a	c	b
b	c	b	c	b	a	c	b	b	c	b	a	b	b	c	b	b	c
	$x(az)$			$(xa)z$			$x(bz)$			$(xb)z$			$x(cz)$			$(xc)z$	

TRIPLE PRODUCT TABLE

A PROOF THAT WOULD PLEASE N. D. KAZARINOFF

C. Stanley Ogilvy, Hamilton College

Let fixed circles of radius m and n, $m>n$, with centers at O and Q respectively, be internally tangent at T. Let a third circle with center P be internally tangent to the larger fixed circle and externally tangent to the smaller. Then $OP+PQ=m+n$, constant for all P. Thus P describes an ellipse with foci at O and Q and a vertex at T. The semi-major axis $a=(m+n)/2$, $c=\frac{1}{2}OQ=(m-n)/2$, and therefore the semi-minor axis $b=\sqrt{(a^2-c^2)}=\sqrt{(mn)}$.

All this is well known. (See for instance Amer. Math. Monthly, 54 (1947) 547.) What has apparently not been mentioned is that it constitutes an elegant if somewhat roundabout proof that the arithmetic mean of two unequal positive numbers is always greater than their geometric mean: $(m+n)/2>\sqrt{(mn)}$.

BOOK REVIEWS

Edited by Dmitri Thoro, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Exploring Mathematics on Your Own. Recent titles in a series of 18 booklets, 64 pages each. *Curves in Space*, *Probability and Chance*, and *Logic and Reasoning in Mathematics.* By Donovan A. Johnson. *Geometric Constructions*, *Basic Concepts of Vectors*, and *Finite Mathematical Systems.* By M. Scott Norton. Webster, St. Louis, 1963, $.92 each (paper).

These booklets are admirably assembled, whetting the mathematical appetite with a soupçon of selected topics. They are most appropriate for beginning college students, potential teachers, and high school mathematics club advisors. Some of the gravamina follow. The word numeral should be substituted for number. There is no mention of the Fibonacci triangle in *Probability and Chance.* Notably absent also, is any discussion of Francis Galton's work on probability; no distinction was made between Poisson distribution and normal distribution.

In *Finite Mathematical Systems*, the presentation of Abelian groups relates quite well to investigations of mathematical learning presently undertaken, notably that by Z. P. Dienes of the Adelaide Mathematics Project in his logic for young children. No triangle-trapezoid-triangle was given in *Geometric Constructions.* Unusual notation was used in *Finite Mathematical Systems*; the ? was used in lieu of the equivalence notation.

In general, the definitions were concise, the presentation straightforward. Of particular merit were the sections on finite geometries, Euler circles, and Peano postulates. The only typographical error I was able to detect was "galatic" for "galactic" on page 14 of *Curves in Space.*

L. V. Rogers, Pinewood School, Los Altos Hills, California

Continued Fractions. By C. D. Olds. New Mathematical Library, Random House, New York, 1963. 162 pp. $1.95.

The subject of continued fractions is allotted a chapter in many books on the theory of numbers. The presentation in such works is condensed and, perhaps because of space limitations, rather restricted. This book by Olds, on the other hand, not only gives an unhurried development of the subject but also explores many ramifications of this significant topic. In addition, the book abounds in historical references giving the sources of the ideas elaborated. The prerequisites for the study of this monograph are kept to a minimum, so that the subject of continued fractions is made available to the ablest of the high school seniors.

Chapter 1 gives the expansion of a rational number as a simple continued fraction. First a wide variety of examples is provided to give the reader a clear picture of the method involved in the expansion. This is followed by a careful presentation of the algebraic structure of the continued fraction in the general rational case. Several properties of convergents, and differences of consecutive convergents, are proved in detail; others are left to the reader as exercises.

Chapter 2 offers an application of continued fractions to the solution of the Diophantine equation $ax+by=c$, where a, b, and c are integers. This equation, if it has any solutions in integers, can be related to an equation of the form $Ax+By=1$ with relatively prime coefficients A and B. The general solution of this equation is readily obtained from any one solution, and one solution can be written at once from the second last convergent of the continued fraction expansion of A/B. The method is satisfying because it is constructive: an actual solution is obtained, not just the information that a solution exists.

Chapter 3 extends to irrational numbers the method of expansion into continued fractions. This necessitates a discussion of limiting processes, of course, and the author does an excellent job of explaining this for his prospective audience of not-too-sophisticated readers. And if the reader should find the analysis a little on the abstruse side, there is also provided the geometric interpretation of Felix Klein. In brief, this is the polygonal approximation to the line $y=\alpha x$ by the points (p_n, q_n) derived from the convergents p_n/q_n to the irrational number α.

Chapter 4 explores the *periodic* continued fraction expansions. It is established that these pertain to quadratic irrational numbers. This theory is applied to the Pell equation $x^2-Ny^2=\pm 1$ by use of the continued fraction expansion of $\sqrt{N}$. The idea is to use the expansion to obtain a smallest positive solution x_0, y_0 and then to derive all solutions from the powers of $x_0+y_0\sqrt{N}$.

Chapter 5 has the title "Epilogue." The topic is the approximation of irrational numbers by rationals, the basis for such a study having been given in Chapter 3. Without giving all proofs, the author leads up to the theorem of Hurwitz, the Markoff chain, and asymmetric approximations. The theorem of Hurwitz states that for any irrational α there are infinitely many rationals h/k such that $|\alpha-h/k|<(\sqrt{5}k^2)^{-1}$, and furthermore that $\sqrt{5}$ is the best possible constant. This result is a gateway to the theory of Diophantine approximations.

This book opens up the theory of continued fractions to a wide class of readers who would find the advanced treatises forbiddingly inaccessible. Exceptionally lucid and well-written, it takes its place among the best expository mathematical works, those that are not only clear but also reliable.

IVAN NIVEN, University of Oregon

Basic Topics in Mathematics. By John Riner. Prentice-Hall, Englewood Cliffs, N. J., 1963. 279 pp. $6.95.

Stated objective: "imparting . . . an awareness of the nature of mathematics" by " . . . a chance to do a little mathematics." And, indeed, little mathematics is done. The initial flame of rigor flickers warningly on page 6 and is out by page 61. Roughly 70 per cent of the exercises are routine; less than half the remaining contribute to the theories presented.

Chapter titles belie their contents: *Algebra* means the reals as an ordered field, *Vector Spaces* means R^2 and R^3, *Limits* means linear polynomials (the one nonlinear proof, $x^2 \to 9$ as $x \to 3$, is pure magic to the tyro) and infinite sequences. Occasionally the author reads the reader's mind ("The idea . . . in most readers' minds . . . ") as a point of expository departure, befuddling one not equipped with the right wrong idea. Once a distance is defined in terms of an undefined distance with no hint that the two are conceptually and, maybe, numerically different, *and* the same symbol is used for both. A well-defined set is so defined that "the collection of all people now living who will be president of the United States after 1961" is not well-defined (although, I presume, it will be eventually). Metric spaces are used as the setting of limits, but no function (except the discrete one) is shown to be a metric.

That this book is written in the latest mathematico-educational jargon and over-burdened with symbolism doesn't prevent its being a cook-book. There are some new recipes, but the old ones are fewer and harder to follow.

There are many typographical and some logical errors.

PAUL YEAROUT, Brigham Young University

Basic Statistics. By Thomas E. Kurtz. Prentice-Hall, Englewood Cliffs, N. J., 1963. $7.50.

The author has written this book to follow a semester of the Kemeny, *et al.*, treatment of probability and a semester of calculus. He asserts in his introduction that "we have our cake and eat it too"; i.e., the book has mathematical content with this limited prerequisite. Bayesian statisticians and believers of adages should bet that he has not been completely successful.

The book gives the impression of having been hastily written and poorly edited. There are a number of general principles which are meaningless or have counter-examples, e.g., "the greater the 'spread' of the values of a random variable, the greater the variance, and vice versa." The following quotations should adequately demonstrate the need for immediate revision. All but the second are displayed in boxes. "A random variable is a function that assigns to each element in the sample space a number," (p. 70). "We may consider the standardized

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form of a random variable as being the same as the original random variable but with its values transformed," (p. 97). "An estimator is a random variable that is used for providing an estimate," (p. 104). "An estimate is the observed value of an estimator," (p. 104). "A statistic is a random variable that depends only on the observed sample," (p. 122). And then on page 152 matrix-valued random variables arise without comment.

Still, a revised version might be quite useful. Honest theorems generally have complete and careful proofs. Expected value is defined as a sum over the sample space with other forms given as theorems. This makes it possible, of course, to avoid the difficulties usually encountered in elementary books. Inference receives careful treatment; nonparametric methods are emphasized, including some of Tukey's work that has not been generally available.

H. E. REINHARDT, Montana State University

Convex Figures and Polyhedra. By L. A. Lyusternik. Translated from the Russian by T. Jefferson Smith. Dover, New York, 1963. x+176 pp. $1.50.

Considering the importance of, and the extent of current interest in, the study of convex figures, there is a lamentable paucity of textual material on the subject in English. Very welcome, therefore, is this English translation of the famous 1956 Russian work by Lyusternik.

The first three chapters, with a little tough sledding in Chapter 3, are readable by a good high school student of mathematics. Here one finds, in addition to basic concepts, elementary proofs of Barbier's theorem on convex figures of constant breadth, the polygonal case of Minkowski's theorem on maximal centrally symmetric bodies on an integral lattice, Cauchy's polyhedral theorem, and Steinitz's fundamental theorem of the theory of polyhedra. Many elegant allied results are stated without proof. Chapter 4, which is more difficult to read, concerns itself with linear systems of convex bodies, planar sections of convex bodies, the Brunn-Minkowski inequality and its consequences. Chapter 5, which utilizes the material of Chapter 4 and which in the Russian edition was written by A. D. Alexandrov, contains a lengthy but elementary proof of the remarkable theorem of Minkowski that a convex polyhedron is determined by the areas and directions of its faces. The Minkowski theorem is established as a corollary to a more general theorem of Alexandrov concerning a condition for two convex polyhedra to be equal and parallel. The sixth, and concluding, chapter of the book is concerned with regular polyhedra, semi-regular convex polyhedra, the isoperimetric problem, chords of arbitrary continua, Blichfeldt's theorem, and a number of other related topics.

It is interesting to note here that Lyusternik's book has essentially little overlap with the fine *Convex Figures* by Yaglom and Boltyanskii (translated into English by P. J. Kelly and L. F. Walton, and published by Holt, Rinehart and Winston in 1961). These two books taken together furnish an excellent elementary introduction to the study of convex figures and bodies in general, and to the contributions of the Russian school in particular.

HOWARD EVES, University of Maine

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Geometric Dissections. By Harry Lindgren. Van Nostrand, Princeton, N. J., 1964. ix+165 pp. $4.95.

Dissection theory is a subject of great theoretical and recreational interest, for it plays a fundamental role in a rigorous development of area and volume and it also yields a seemingly inexhaustible supply of attractive and challenging puzzles. The work under review is devoted entirely to the recreational aspect of the subject. It is probably the only book-length treatment to be found in any language, and it is written with charm and skill by an expert dissectionist whose contributions have appeared in the puzzle corners of magazines and newspapers the world over.

Not all facets of the recreational aspect of dissection theory are considered, for the author pretty much limits himself to the problem of dissecting one figure into another with as few pieces as possible. Though the minimal dissection problem has been solved with certainty in only a very few instances, it seems unlikely that some of the ingenious systematic approaches devised by the author can be improved upon. The book is easy to read, with appeal to both amateur and professional. It contains over 400 dissection puzzles (mostly planar, but some spatial), a set of 56 dissection problems on which the reader is invited to test his strength before peeking at the solutions in the back of the book, some 17 pages of excellent drawings of polygons, strips, and tesselations that the reader can carefully trace and use in experimental fashion, and some tables of numerical dimensions for the convenience of those who wish to construct accurate drawings. The book is profusely and elegantly illustrated, and can without doubt furnish a puzzle enthusiast endless hours of enjoyable pursuit.

HOWARD EVES, University of Maine

BRIEF MENTION

A History of Geometrical Methods. By Julian Lowell Coolidge. Dover, New York, 1963. xv+455 pp. $2.25.

Republication of the work first published by the Oxford University Press in 1940.

The Mathematics of Great Amateurs. By Julian Lowell Coolidge. Dover, New York, 1963. vii+210 pp. $1.50.

Republication of the work first published by the Oxford University Press in 1949.

A History of the Mathematical Theories of Attraction and the Figure of the Earth. By I. Todhunter. Dover, New York, 1963. 508 pp. $7.50.

Republication of the work first published in 1873.

A Manual of Greek Mathematics. By Sir Thomas L. Heath. Dover, New York, 1963. 552 pp. $2.25.

Republication of the work first published in 1931.

Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections. By Karl Friedrich Gauss. Translated by Theoria Motus. Dover, New York, 1963. 326 pp. $2.95.

Republication of the work first published in 1857.

Mathematics. By David Bergamini and the editors of *Life*. Time Inc., New York, 1963. 200 pp. $3.95.

Geometric Dissections. By Harry Lindgren. Van Nostrand, Princeton, N. J., 1964. ix+165 pp. $4.95.

Dissection theory is a subject of great theoretical and recreational interest, for it plays a fundamental role in a rigorous development of area and volume and it also yields a seemingly inexhaustible supply of attractive and challenging puzzles. The work under review is devoted entirely to the recreational aspect of the subject. It is probably the only book-length treatment to be found in any language, and it is written with charm and skill by an expert dissectionist whose contributions have appeared in the puzzle corners of magazines and newspapers the world over.

Not all facets of the recreational aspect of dissection theory are considered, for the author pretty much limits himself to the problem of dissecting one figure into another with as few pieces as possible. Though the minimal dissection problem has been solved with certainty in only a very few instances, it seems unlikely that some of the ingenious systematic approaches devised by the author can be improved upon. The book is easy to read, with appeal to both amateur and professional. It contains over 400 dissection puzzles (mostly planar, but some spatial), a set of 56 dissection problems on which the reader is invited to test his strength before peeking at the solutions in the back of the book, some 17 pages of excellent drawings of polygons, strips, and tesselations that the reader can carefully trace and use in experimental fashion, and some tables of numerical dimensions for the convenience of those who wish to construct accurate drawings. The book is profusely and elegantly illustrated, and can without doubt furnish a puzzle enthusiast endless hours of enjoyable pursuit.

HOWARD EVES, University of Maine

BRIEF MENTION

A History of Geometrical Methods. By Julian Lowell Coolidge. Dover, New York, 1963. xv+455 pp. $2.25.

Republication of the work first published by the Oxford University Press in 1940.

The Mathematics of Great Amateurs. By Julian Lowell Coolidge. Dover, New York, 1963. vii+210 pp. $1.50.

Republication of the work first published by the Oxford University Press in 1949.

A History of the Mathematical Theories of Attraction and the Figure of the Earth. By I. Todhunter. Dover, New York, 1963. 508 pp. $7.50.

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Republication of the work first published in 1857.

Mathematics. By David Bergamini and the editors of *Life.* Time Inc., New York, 1963. 200 pp. $3.95.

Philosophy of Mathematics: Selected Readings. By Paul Benacerraf and Hilary Putnam, editors. Prentice-Hall, Englewood Cliffs, N. J., 1964. vii+536 pp. $8.95.

This extensive collection of essays by philosophers and mathematicians presents many points of view on the nature of mathematics. There are four parts: Foundations of Mathematics, The Existence of Mathematical Objects, Mathematical Truth, Wittgenstein on Mathematics.

An Introduction to the History of Mathematics, revised edition. By Howard Eves. Holt, Rinehart, and Winston, New York, 1964. xvi+439 pp. $7.95.

Philosophical Problems of Space and Time. By Adolf Grunbaum. Alfred A. Knopf, New York, 1964. 450 pp. $10.75.

Applied Calculus. By L. J. Adams. Wiley, New York, 1963. ix+278 pp. $5.95.

Basic Mathematics for General Education. 3rd ed. By Harold C. Trimble, E. W. Hamilton and Ina Mae Silvey. Prentice-Hall, Englewood Cliffs, N. J., 1963. xi+332 pp. $6.00.

Revision of the 1955 edition.

The Japanese Abacus Explained. By Y. Yoshino. Dover, New York, 1963. xiii+240 pp. $1.25.

Republication of the original 1937 edition with a new introduction by Martin Gardner.

Advanced Abacus: Japanese Theory and Practice. By Takashi Kojima. Charles E. Tuttle, Rutland, Vermont. 159 pp. $2.25.

A sequel to the author's *The Japanese Abacus: Its Use and Theory.*

Fallacies in Mathematics. By E. A. Maxwell. Cambridge University Press, New York, 1963. 95 pp. $.95.

First paperback edition of the 1959 publication.

ANSWERS

A354. The greatest angle is subtended at the point of contact of the smaller of two circles through A and B which touch the given circle internally. The other points on the same side of AB give smaller angles as they are outside the touching circle, while the points on the other side give smaller angles because they are outside the larger touching circle.

A355. Multiply the product by $(1/2)(3^{2^0}-1)$ or 1 and obtain $(1/2)(3^{2^{n+1}}-1)$ since $(3^{2^0}-1)(3^{2^0}+1)=(3^{2^1}-1)$, $(3^{2^1}-1)(3^{2^1}+1)=(3^{2^2}-1)$, and so on.

In general, if any base $x>1$ were used instead of 3, the product would be

$$[1/(x-1)](x^{2^{n+1}}-1).$$

A356. The leg of a right triangle is shorter than the hypotenuse. Consider the right triangle with legs of length 1 and x.

A357. Let $S(1)=2$, $S(2)=3$, $\cdots$.

Then since $S(a)=2+\sum_{i=0}^{a-1} i$ and $S(a)-S(b)=\sum_{i=b}^{a-1} i$ for $a>b$, we have $S(501)=125252$ and $S(501)-S(500)=500$.

A358. In any base B, $2(297)<2(300)=600$. $600<792<800=4(200)<4(297)$. Hence $792=3(297)$ so that $7B^2+9B+2=6B^2+27B+21$ or $B^2-18B-19=0$. Rejecting the negative root leaves $B=19$.

(Quickies on page 124)

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

579. *Proposed by David L. Silverman, Beverly Hills, California.*

"If two of my children are selected at random, likely as not, they will be of the same sex," said the Sultan to the Caliph. "What are the chances that both will be girls?" asked the Caliph. "Equal to the chance that one child selected at random will be a boy," replied the Sultan. How many children did he have?

580. *Proposed by Joseph Arkin, Spring Valley, New York.*

Is a solution in integers possible for the equation $(c-a-b)^3=24\,abc$, where a, b and c are not zero?

581. *Proposed by Joseph L. Teeters, Baker University, Kansas.*

If a complex number $a+bi$ is defined

I. to be positive when (i) $b>0$ or (ii) $b=0$ and $a>0$
II. to be zero when $b=0$ and $a=0$, and
III. to be negative when (i) $b<0$ or (ii) $b=0$ and $a<0$

and if $A<B$ (A, B being complex) means that $B-A$ is positive, then prove or disprove the following:

1. If A, B, C are complex numbers, and $A<B$, then $A+C<B+C$.
2. If A, B, C are complex numbers, and $A<B$, and C is positive, then $AC<BC$.

582. *Proposed by Charles W. Trigg, San Diego, California.*

A regular octahedron, edge e, is cut by a plane parallel to one of its faces. Find:

(a) the perimeter, and
(b) the area of the section.

583. *Proposed by Kaidy Tan, Fukien Normal College, Fukien, China.*

Solve a rational triangle (i.e., the lengths of each side is a rational number) so that the altitude, the median, and the angle bisector on one side are rational numbers.

584. *Proposed by J. Barry Love, Eastern Baptist College, Pennsylvania.*

Sum the series for $|x|>1$,

$$\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\frac{8}{x^8+1}+\cdots.$$

585. *Proposed by J. M. Howell, Los Angeles City College.*

A population consists of three types of objects. Let P, Q and R represent the probability of drawing one of each type on a single draw, $P+Q+R=1$. A sample of size n is drawn with replacement, and p, q, r represent the fractions of these three types of objects found in the sample. The mean and variance of $p+r$ would be $P+R$ and $(P+R)Q/n$, since this would be a binomial distribution. What are the mean and variance of $p-r$?

SOLUTIONS

Late Solutions

Claudia Abbott, University of California at Riverside: **551.** *John A. Burslem, St. Louis University*: **551, 552, 553, 554, 556.** *Larry Hoehn, Perryville, Missouri*: **554.** *Abraham Karen, Teaneck, New Jersey*: **551, 552, 554, 556, 557.** *F. W. Lovsin, Brantford, Ontario, Canada*: **545.** *M. G. Murdeshwar, University of Alberta*: **538, 539, 540, 542, 543, 544.** *Shri K. C. Sharma, Institute of Armament Technology, Dapodi, Poona, India*: **545, 550.** *Polly Spital, California State Polytechnic College*: **544.** *Sidney Spital, California State Polytechnic College*: **538, 539, 540, 542, 543, 544, 545, 547, 548, 549.** *Sister M. Stephanie, Georgian Court College, New Jersey*: **552, 544.** *Howard L. Walton, Falls Church, Virginia*: **544.**

Comment on Problem 516

516. [May, 1963 and January, 1964] *Proposed by Maxey Brooke.*

Comment by G. H. L. Buxton, National Engineering Laboratory, East Kilbride, Lanarkshire, Scotland.

The problem is indeterminate and the solution published is incorrect.

From statements 1, 3, 4 and 6, D is not Adams, McCall, Jones or Williams, and from 4 and 5, not Smith. Therefore, D is Brown. This leaves X or T for Adams or Williams, and from 3, X is Adams and T is Williams. Therefore from 5, Q is Smith and Y is McCall, and the murderer is either Smith or McCall.

Condition 2 is extraneous as stated. If Condition 2 were replaced by "The murderer has no relatives," or Condition 5 replaced by " . . . sits to the left of the murderer," the problem would be determinate and the solution given correct.

Editor's note. The difference between this interpretation and that of all but one of the other solvers of this problem apparently lies in the interpretation of Condition 4. If a person is included in the set of those who do not sit next to him, then Buxton's solution follows. If he is excluded from the set, the other solutions hold.

Errata

In the solution to Problem 489, Page 135, of Volume 36, Number 2, November, 1963, all the expressions appearing as $a\sqrt{b}/c$ should read $(a\sqrt{b})/c$.

Fair Game

558. [September, 1964] *Proposed by David L. Silverman, Beverly Hills, California.*

Players A and B each has a die, which he places as he sees fit on a table top, without seeing his opponent's play. Simultaneously, the two dice are shown, and the total of the upper faces determines the winner. A wins if the total is a prime; B wins otherwise. Over a long period of time, whom does the game favor?

I. *Solution by Maxey Brooke, Sweeny, Texas.*

1. If both play at random, A's odds are 15/36.
2. If A plays only 1's and B plays at random, A's odds are 2/3.
3. If A plays only 2's and B plays at random, A's odds are 1/2.
4. If A plays 3, 4, 5 or 6 and B plays at random, A's odds are 1/3.
5. Hence A's optimum game is to play 1 or 2 at random. If B then plays at random, A's odds are 7/12.
6. But B's optimum game will be to play 3, 4, 5, 6 at random.

If both play their optimum game, A's odds are 1/2 and the game favors neither.

II. *Solution by W. W. Funkenbusch, Michigan Technological University.*

We assume that the dice used are regular polyhedra. We eliminate the tetrahedron since it does not have an "upper face" when placed down. Elementary game theory shows that the game is fair if the die is a cube, otherwise the game favors B.

More specifically if the payoff is 1 if A wins and -1 if B wins, we find:

(a) *for cube*: A's mix consists of equal parts of 1 and 2. B's mix consists of equal parts of 3 and 4. The game value is 0.

(b) *for octahedron*: A's mix consists of equal parts of 1, 2, 3, 4, 5 and 6. B's mix consists of equal parts of 3, 4, 5, 6, 7 and 8. The game value is $-1/3$.

(c) *for dodecahedron*: A's mix consists of equal parts of 1, 2, 3, 4, 5 and 6. B's mix consists of equal parts of 3, 4, 5, 6, 7 and 8. The game value is $-1/3$.

(d) *for icosahedron*: A's mix consists of unit parts of 1, 2, 5, 6, 11 and 12, and double parts of 3 and 4. B's mix consists of unit parts of 13, 14, 15, 16, 19 and 20. and double parts of 17 and 18. The game value is $-1/2$.

Also solved by Leon Bankoff, Los Angeles, California; Merrill Barneby, University of North Dakota; Dermott A. Breault, Sylvania ARL, Waltham, Massachusetts; John A. Burslem, St. Louis University; Ralph L. Carmichael, NASA, Moffett Field; Harry M. Gehman, SUNY at Buffalo, New York; Carl Harris, Polytechnic Institute of Brooklyn; J. A. H. Hunter, Toronto, Ontario, Canada; Robert F. Jackson, University of Toledo; Richard A. Jacobson, South Dakota State University; Robert Lera, St. Mary's College, California; Robert Martel, St. Mary's College, California; Prasert Na Nagara, Bangkok, Thailand; Benjamin Sharpe, SUNY at Buffalo, New York; J. S. Vigder, Defence Research Board of Canada; and the proposer.

Several solvers imposed the assumption that the players would use the numbers which appeared as the dice were rolled: *Len Bertain, St. Mary's College, California; Ralph L. Carmichael, Moffet Field, California; Sidney Spital, California State Polytechnic College; C. W. Trigg, San Diego, California; Ralph N. Vawter, St. Mary's College, California; and Benjamin B. Winter, Autonetics, Anaheim, California.*

Improper Integral

559. [September, 1964] *Proposed by Gilbert Labelle, Université de Montréal.*

Show that

$$\int_0^\infty \frac{dx}{1+x^\pi} = \frac{1}{\sin 1}.$$

Solution by Raymond E. Whitney, Lock Haven State College, Pennsylvania.

Consider $I=\int_0^\infty dx/(1+x^{1/p})$; $p\in(0,1)$ or its equivalent, $q=1/p$,

$$\int_0^\infty dx/(1+x^q);\ q\in(1,\infty).$$

With the transformation $x^{1/p}=x^q=u$,

$$I = p\int_0^\infty u^{p-1}du/(1+u) = 1/q\int_0^\infty u^{1/q-1}du/(1+u).$$

Now the given integral is a well-known contour integral (*A Course of Modern Analysis*, Whittaker & Watson, pp. 117, 118 (1961)) and has the value $\pi\csc p\pi = \pi\csc\pi/q$.

Hence $I=p\pi\csc p\pi=\pi/q\csc\pi/q$.

Substitution of $p=1/\pi$ or $q=\pi$ in the above yields the desired result.

Also solved by Jacques Allard, University of Sherbrooke, Canada; Winifred Asprey, Vassar College; John A. Burslem, St. Louis University; Len Bertain, St. Mary's College, California; J. D. Cloud, Manhattan Beach, California; Sidney Glusman, New York, New York; Carl Harris, Polytechnic Institute of Brooklyn; John E. Hosmer, Jr., Wisconsin State College, La Crosse; Robert F. Jackson, University of Toledo; Otto J. Karst, Webb Institute of Naval Architecture; Peng Aun Khor, Queen's University, Kingston, Ontario, Canada; Robert Lera, St. Mary's College, California; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Robert Martel, St. Mary's College, California; Henry J. Ricardo, Yeshiva University, New York; Sidney Spital, California State Polytechnic College; William Squire, West Virginia University; M. N. Srikanta Swanny, University of Saskatchewan, Regina, Canada; Ralph N. Vawter, St. Mary's College, California; Dale Woods, Northeast Missouri State Teachers College; and the proposer.

References to this problem were found in a number of well-known books.

Inscribed Triangles

560. [September, 1964] *Proposed by Morton Hackman, University of Washington.*

Show that the perimeter of a triangle inscribed in a circle is at least twice the diameter of the circle if the triangle contains the center of the circle.

I. *Solution by John Selfridge, University of Washington.*

Let the triangle have vertices A, B, C. At least two sides of the triangle, say AB and AC, must subtend arcs $\geqq 90°$. Let AD be the diameter through A. Because the center of the circle must lie in the triangle, B must lie on one side of AD, and C must lie on the other side. Let BC intersect AD at P, and let BQ be the perpendicular dropped from B to AD. Then $AB \geqq AQ$ because it is the hypotenuse of a right triangle. For the same reason $BP \geqq BQ$. But $BQ \geqq QD$ because the triangle BQD is similar to the triangle ABD, and the arc subtended by AB, being $\geqq 90°$, is at least as large as the arc subtended by BD. Then $AB+BP \geqq AQ+QD=AD$. In similar fashion, $AC+CP \geqq AD$, so that $AB+AC+BC \geqq 2AD$.

II. *Solution by Dermott A. Breault, Sylvania ARL, Waltham, Massachusetts.*

Inscribe a triangle with sides A, B and C in a unit circle, and label the central angles subtended by them a, b, and c respectively. If the center falls in or on the triangle, we will have:

$$a + b + c = 360°, \tag{1}$$

$$a + b \geqq 180°. \tag{2}$$

Three applications of the law of cosines gives:

$$a^2 + b^2 + c^2 = 2((1 - \cos a) + (1 - \cos b) + (1 - \cos c)), \tag{3}$$

from which, with the aid of (1), we arrive at

$$a + b + c = 2(\sqrt{(1 - \cos a)} + \sqrt{(1 - \cos b)} + \sqrt{(1 - \cos (a + b))}). \tag{4}$$

Keeping (2) in mind, we find that (4) takes on a maximum value of $3\sqrt{3}$ when $a=b=120°$, and a minimum value of 4 for $a=0°$, $b=180°$.

Also solved by Merrill Barneby, University of North Dakota; John A. Burslem, St. Louis University; Thomas V. Eynden, Kodiak, Alaska; Philip Fung, Fenn College, Ohio; Michael Goldberg, Washington, D. C.; Neal Harrell, Menlo-Atherton High School, Atherton, California; Larry Hoehn, Perryville, Missouri; Robert F. Jackson, University of Toledo; Richard A. Jacobson, South Dakota State University; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; Prasert Na Nagara, Bangkok, Thailand; Stanley Rabinowitz, Far Rockaway, New York; Lawrence A. Ringenberg, Eastern Illinois University; Sidney Spital, California State Polytechnic College; J. S. Vigder, Defence Research Board of Canada; and Dale Woods, Northeast Missouri State Teachers College.

A Fibonacci Property

561. [September, 1964] *Proposed by Benjamin B. Sharpe, State University of New York at Buffalo.*

Prove that $a^2+b^2=c^2$, $(a \leqq b < c)$ is impossible if a, b, and c are Fibonacci numbers.

I. *Solution by Sidney Spital, California State Polytechnic College.*

The Fibonacci sequence is generated by the recurrence relation $\sigma_n=\sigma_{n-1}+\sigma_{n-2}$. We square this equation and partially transpose:

$$\sigma_n^2 - \sigma_{n-1}^2 = \sigma_{n-2}^2 + 2\sigma_{n-2}\sigma_{n-1}.$$

But since the sequence is a monotone increasing one,

$$2\sigma_{n-2} > \sigma_{n-2} + \sigma_{n-3} = \sigma_{n-1}, \quad \text{giving} \quad \sigma_n^2 - \sigma_{n-1}^2 > \sigma_{n-1}^2.$$

Therefore,

$$\sigma_n^2 = \sigma_k^2 \geqq \sigma_n^2 - \sigma_{n-1}^2 > \sigma_m^2$$

and

$$\sigma_n^2 > \sigma_k^2 + \sigma_m^2 \quad \text{for} \quad k, m = 1, 2, \cdots, n-1.$$

This precludes the possibility of three of them satisfying $c^2=a^2+b^2$, $a \leqq b < c$.

II. *Solution by Clifton T. Whyburn, Louisiana State University.*

Suppose $a^2+b^2=c^2$, $a \leqq b < c$ has a solution in Fibonacci numbers u_α, u_β, u_γ. Then $u_\alpha \leqq u_\beta \leqq u_{\gamma-1}$ and $2u_{\gamma-1}^2 \geqq u_\gamma^2$, $\sqrt{2} \geqq u_\gamma/u_{\gamma-1}$. This, however, cannot be true for any $\gamma > 1$, as is shown by induction: $u_2/u_1=1$; assume

$$\frac{u_k}{u_{k-1}} \leqq \sqrt{2}, \qquad \frac{u_{k-1}}{u_{k-2}} > \sqrt{2}.$$

Then $\sqrt{2} \geqq \dfrac{u_k}{u_{k-1}} = 1 + \dfrac{u_{k-2}}{u_{k-1}} > 1 + \dfrac{1}{\sqrt{2}}$ which is not true.

Also solved by Joseph Arkin, Spring Valley, New York; Merrill Barneby, University of North Dakota; Leon Bankoff, Los Angeles, California; Dermott A. Breault, Sylvania ARL, Waltham, Massachusetts; John L. Brown, Jr., Pennsylvania State University; L. Carlitz, Duke University; Darel W. Hardy, Seattle University; Robert F. Jackson, University of Toledo; Richard A. Jacobson, South Dakota State University; Sidney Kravitz, Dover, New Jersey; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Burton Navid and Emanuel Vegh (jointly), U. S. Naval Research Laboratory, Washington, D. C.; Stanley Rabinowitz, Far Rockaway, New York; David L. Silverman, Beverly Hills, California; Lurline S. Squire, Morgantown, West Virginia; C. W. Trigg, San Diego, California; G. Joseph Wimbish, Jr., University of Oklahoma; and the proposer. J. D. E. Konhauser found this problem as E1028 [March, 1953], Amer. Math. Monthly, 60 (1953) 191.

Carlitz pointed out that this result is proved in *A Note on Fibonacci Numbers*, The Fibonacci Quart. 2 (1964) 15–28, Theorem 5. The result holds for both the Fibonacci and Lucas numbers. More generally (Theorem 6) it is proved that the equations

$$\overset{r}{u}_m + \overset{r}{u}_n = \overset{r}{u}_k \qquad (0 < m \leqq n),$$
$$\overset{r}{v}_m + \overset{r}{v}_n = \overset{r}{v}_k \qquad (0 \leqq m \leqq n),$$

are impossible for all $r \geqq 2$, where

$$u_0 = 0,\ u_1 = 1, \qquad u_{n+1} = u_n + u_{n-1} \qquad (n \geqq 1),$$
$$v_0 = 2,\ v_1 = 1, \qquad v_{n+1} = v_n + v_{n-1} \qquad (n \geqq 1).$$

Angles in a Hexagon

563. [September, 1964] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let A, B', A', B be four consecutive vertices of a regular hexagon. If M is an arbitrary point of the circumcircle (in particular on arc $A'B'$) and MA, MB intersect BB' and AA' in the points E and F respectively, then prove that:

$$\text{(a)} \quad \measuredangle MEF = 3\measuredangle MAF$$
$$\text{(b)} \quad \measuredangle MFE = 3\measuredangle MBE.$$

Solution by Richard A. Jacobson, South Dakota State University.

Let $AB' = x$ and $\measuredangle MAF = a$. Noting that $\measuredangle AB'B = \measuredangle AA'B = \measuredangle AMB = 90°$, we have from triangles AMB, $AB'E$ and $AA'B$ that $AM = 2x\cos(30+a)$, $BM = 2x\sin(30+a)$, $AE = x/\cos(30-a)$ and $BF = x/\cos a$. Thus in triangle EMF we find that

$$\tan(\measuredangle MEF) = \frac{MF}{ME} = \frac{BM - BF}{AM - AE} = \frac{2x\sin(30+a) - \dfrac{x}{\cos(a)}}{2x\cos(30+a) - \dfrac{x}{\cos(30-a)}}$$

$$= \frac{\dfrac{2\sin(30+a)\cos(a) - 1}{\cos a}}{\dfrac{2\cos(30+a)\cos(30-a) - 1}{\cos(30-a)}}$$

$$= \frac{2\sin(30+2a) - 1}{\cos(a)} \cdot \frac{\cos(30-a)}{2\cos(2a) - 1}$$

$$= \frac{2\sin(30+2a)\cos(30-a) - \cos(30-a)}{2\cos(2a)\cos(a) - \cos(a)}$$

$$= \frac{\sin(60+a) + \sin(3a) - \cos(30-a)}{\cos(3a) + \cos(a) - \cos(a)}$$

$$= \frac{\sin(3a)}{\cos(3a)} = \tan(3a).$$

Since $a \leqq 30°$, we have $\measuredangle MEF = 3\measuredangle MAF$. Part (b) is done similarly.

Also solved by Leon Bankoff, Los Angeles, California; J. D. E. Konhauser, HRB-Singer, State College, Pennsylvania; Stanley Rabinowitz, Far Rockaway, New York; Sidney Spital, California State Polytechnic College; and the proposer.

Gamma Function

564. [September, 1964] *Proposed by Murray R. Spiegel, Rensselaer Polytechnic Institute.*

Show that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - 1/2\sin^2\theta)}} = \frac{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14\cdot 16 \cdots}{5\cdot 5\cdot 9\cdot 9\cdot 13\cdot 13\cdot 17\cdot 17 \cdots}.$$

Solution by Leonard Carlitz, Duke University.

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - 1/2\sin^2\theta)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - 1/2\cos^2\theta)}} = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 + \sin^2\theta)}}$$

$$= \sqrt{2}\int_0^1 \frac{dx}{\sqrt{(1 - x^4)}} = \frac{\sqrt{2}}{4}\int_0^1 t^{-3/4}(1 - t)^{-1/2}\,dt - \frac{\sqrt{2}}{4}\,B(1/4, 1/2)$$

$$= \frac{\sqrt{2}}{4}\,\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)} = \frac{\sqrt{2}}{4}\,\frac{\Gamma^2(1/4)\Gamma(1/2)}{\dfrac{\pi}{\sin\dfrac{\pi}{4}}} = \frac{1}{4\sqrt{\pi}}\Gamma^2(1/4).$$

On the other hand,

$$\lim_{n=\infty} \frac{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12 \cdots (4n - 2)(4n)}{1\cdot 5\cdot 5\cdot 9\cdot 9\cdot 13\cdot 13 \cdots (4n + 1)(4n + 1)}$$

$$= \prod_{n=1}^{\infty} \frac{n(n - 1/2)}{(n + 1/4)(n + 1/4)} = \frac{\Gamma(1 + 1/4)\Gamma(1 + 1/4)}{\Gamma(1)\Gamma(1/2)}$$

(See for example Whittaker and Watson, Modern Analysis, 4th Edition, Page 239.)

$$= \frac{\Gamma^2(1/4)}{16\sqrt{\pi}},$$

so that

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{(1 - 1/2\sin^2\theta)}} = \frac{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12 \cdots}{1\cdot 5\cdot 5\cdot 9\cdot 9\cdot 13\cdot 13 \cdots}.$$

Also solved by J. S. Frame, Michigan State University; Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania; and the proposer. One incorrect solution was received. G. J. Wimbish, Jr., University of Oklahoma, pointed out that as stated the product on the right diverges to zero.

Comment on Q342

Q342. [September, 1964] *Comment by Martin S. Erdsneker, Bronx Community College.*

For all integers $N>1$ we have $(N+1)^N>N!$ or $N+1>(N!)^{1/N}$. Multiplying both sides, $N!(N+1)>N!(N!)^{1/N}$ or $(N+1)!>(N!)^{(N+1)/N}$. That is

$$[(N+1)!]^{1/(N+1)} > (N!)^{1/N}$$

which proves that for all $N>1$

$$\sqrt[N+1]{(N+1)!} > \sqrt[N]{N!}$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q354. Given two points A and B inside a circle, at what point on the circumference of the circle does AB subtend the greatest angle?

[*Submitted by Alan Sutcliffe*]

Q355. Simplify the product

$$(3^{2^0}+1)(3^{2^1}+1)(3^{2^2}+1)\cdots(3^{2^n}+1).$$

[*Submitted by C. W. Trigg*]

Q356. Give a geometric argument for the nonexistence of positive solutions to the equation

$$x=\sqrt{(1+x^2)}.$$

[*Submitted by R. G. Buschman*]

Q357. Find the 501st term and the difference between this and the preceding term of the sequence

$$[2, 3, 5, 8, 12, \cdots].$$

[*Submitted by Myron Tepper*]

Q358. In what base is 297 a factor of 792?

[*Submitted by D. L. Silverman*]

(Answers on page 115)

Philosophy of Mathematics: Selected Readings. By Paul Benacerraf and Hilary Putnam, editors. Prentice-Hall, Englewood Cliffs, N. J., 1964. vii+536 pp. $8.95.

This extensive collection of essays by philosophers and mathematicians presents many points of view on the nature of mathematics. There are four parts: Foundations of Mathematics, The Existence of Mathematical Objects, Mathematical Truth, Wittgenstein on Mathematics.

An Introduction to the History of Mathematics, revised edition. By Howard Eves. Holt, Rinehart, and Winston, New York, 1964. xvi+439 pp. $7.95.

Philosophical Problems of Space and Time. By Adolf Grunbaum. Alfred A. Knopf, New York, 1964. 450 pp. $10.75.

Applied Calculus. By L. J. Adams. Wiley, New York, 1963. ix+278 pp. $5.95.

Basic Mathematics for General Education. 3rd ed. By Harold C. Trimble, E. W. Hamilton and Ina Mae Silvey. Prentice-Hall, Englewood Cliffs, N. J., 1963. xi+332 pp. $6.00.

Revision of the 1955 edition.

The Japanese Abacus Explained. By Y. Yoshino. Dover, New York, 1963. xiii+240 pp. $1.25.

Republication of the original 1937 edition with a new introduction by Martin Gardner.

Advanced Abacus: Japanese Theory and Practice. By Takashi Kojima. Charles E. Tuttle, Rutland, Vermont. 159 pp. $2.25.

A sequel to the author's *The Japanese Abacus: Its Use and Theory.*

Fallacies in Mathematics. By E. A. Maxwell. Cambridge University Press, New York, 1963. 95 pp. $.95.

First paperback edition of the 1959 publication.

ANSWERS

A354. The greatest angle is subtended at the point of contact of the smaller of two circles through A and B which touch the given circle internally. The other points on the same side of AB give smaller angles as they are outside the touching circle, while the points on the other side give smaller angles because they are outside the larger touching circle.

A355. Multiply the product by $(1/2)(3^{2^0}-1)$ or 1 and obtain $(1/2)(3^{2^{n+1}}-1)$ since $(3^{2^0}-1)(3^{2^0}+1)=(3^{2^1}-1)$, $(3^{2^1}-1)(3^{2^1}+1)=(3^{2^2}-1)$, and so on.

In general, if any base $x>1$ were used instead of 3, the product would be

$$[1/(x-1)](x^{2^{n+1}}-1).$$

A356. The leg of a right triangle is shorter than the hypotenuse. Consider the right triangle with legs of length 1 and x.

A357. Let $S(1)=2$, $S(2)=3, \cdots$.

Then since $S(a)=2+\sum_{i=0}^{a-1} i$ and $S(a)-S(b)=\sum_{i=b}^{a-1} i$ for $a>b$, we have $S(501)=125252$ and $S(501)-S(500)=500$.

A358. In any base B, $2(297)<2(300)=600$. $600<792<800=4(200)<4(297)$. Hence $792=3(297)$ so that $7B^2+9B+2=6B^2+27B+21$ or $B^2-18B-19=0$. Rejecting the negative root leaves $B=19$.

(Quickies on page 124)